Factorization algebras in perturbative QFT

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Over the last several decades, myriad interactions between physics and higher structures have arisen.

Physicists now use K-theory, derived categories, stacks, operads, (∞, n)-categories, ...

Basic question: *Why?*

Of course, in any particular case, there is a particular answer.

The goal of this talk is to introduce a (partial) explanation of the general phenomenon.

$$\text{QFT} \overset{\text{observables}}{\Longrightarrow} \text{factorization algebras} \overset{\text{fancy stuff}}{\Longrightarrow} \text{higher structures}$$

(perturbative & Euclidean)

$$\Uparrow$$

topology,
representation theory,
⋮

Provisos:

- Applies to *Euclidean* signature (Kasia and I are thinking about possibilities in Lorentzian signature — happy to discuss!)
- Higher structures are best developed for *topological* field theories, so examples are skewed in that direction
- Need to develop methods further to include boundary conditions (cf. lovely work of Cattaneo-Mnëv-Reshetikhin) and defects

Nonetheless, there are examples, & hopefully almost everyone here will find one example suitable for understanding how to think in this framework.

Topological field theories:

People	Theory	Algebra
Grady-Li-Li	1D σ-model into symplectic manifold	Fedosov quantization
Li-Li	topological B-model with target Calabi-Yau X	$\mathcal{E}_2$ algebra deformation of $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X[-1])$
Kontsevich/ Cattaneo-Felder	Poisson σ-model	$\mathcal{E}_2$ algebra in bulk & def. quant. on boundary
Axelrod-Singer/ Kontsevich/...	Chern-Simons theory	$\mathcal{E}_3$ algebra deformation of $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$

Holomorphic field theories:

People	Theory	Algebra
Gorbounov-G.-Williams	curved $\beta\gamma$ system into T^*X	chiral differential operators CDO_X
G.-Williams	holomorphic string (X: CY, string, dim $= 13$)	semi-infinite cohomology of CDO_X
Nekrasov/ Williams	higher dimensional analogs	"higher" vertex algebras
Costello-Li	BCOV theory (also, coupled to holomorphic Chern-Simons theory)	"higher" vertex algebras

"Mixed" field theories: Costello has constructed

- a 4D gauge theory for Lie algebra $\mathfrak{g}$ that recovers a Yangian for $\mathfrak{g}$
- a 5D gauge theory that recovers a quantum group arising as a deformation of $U\mathfrak{g}[z_1, z_2]$

Lurie asserted, and Scheimbauer proved, that a special class of factorization algebras ($\mathcal{E}_n$ algebras) determine fully extended framed n-dimensional TFTs.

Recently, Ayala & Francis have generalized factorization homology, and they recover the full Cobordism Hypothesis as a consequence. (Their work should be well-suited for dealing with defects.)

- Beilinson-Drinfeld's work on the geometric Langlands program, and its offshoots
- Labeled configuration spaces *à la* Segal, McDuff, Salvatore, Lurie, etc
- Gaitsgory-Lurie's proof of the Weil conjecture on quadratic forms on curves in characteristic p fields (ideas extend to characteristic p!)
- Ben-Zvi–Brochier–Jordan's recovery of Alekseev's algebras (e.g., quantum D-modules)

Let M be manifold on which the fields live.

Let $\mathcal{E}$ be the sheaf of fields (e.g., sections of a vector bundle). For $U \subset V$, we get a restriction $\mathcal{E}(V) \to \mathcal{E}(U)$.

Let $\mathcal{O}(\mathcal{E}(U))$ denote "functions on fields on U." I'll call these observables. Then $\mathcal{O}(\mathcal{E}(-))$ is a *co*sheaf of commutative algebras. For $U \subset V$, we get an extension $\mathcal{O}(\mathcal{E}(U)) \to \mathcal{O}(\mathcal{E}(V))$.

When you quantize, however, you stop having a commutative algebra on every open, so the quantized observables $\mathcal{O}^{\mathrm{q}}$ are not an *ordinary* cosheaf. What structure is left on $\mathcal{O}^{\mathrm{q}}$?

Prefactorization algebras

M — topological space (think: smooth manifold)

A *prefactorization algebra* $\mathcal{F}$ on M with values in dg vector spaces is:

- a dg vector space $\mathcal{F}(U)$ for each open set $U \subset M$
- a cochain map $\mathcal{F}(U) \to \mathcal{F}(V)$ for each inclusion $U \subset V$
- a cochain map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for each $U_1, \ldots, U_n \subset V$ with the U_i pairwise disjoint.

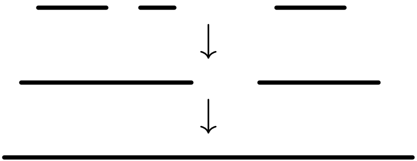
satisfying

- equivariance under relabeling
- associative for composition: if $U_{i,1} \sqcup \cdots \sqcup U_{i,n_i} \subseteq V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subseteq W$, the following diagram commutes:

$$\begin{array}{ccc} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) & \longrightarrow & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

Key example

An associative algebra A determines a prefactorization algebra on $\mathbb{R}$:



$$\begin{array}{ccc} a \otimes b \otimes c & \in & A \otimes A \otimes A \\ \downarrow & & \downarrow \\ ab \otimes c & \in & A \otimes A \\ \downarrow & & \downarrow \\ abc & \in & A \end{array}$$

There is a colored operad $\mathrm{Disj}(M)$ whose colors are open sets in M and which has precisely one operation $U_1, \ldots, U_n \to V$ if U_i are pairwise disjoint and contained in V.

A prefactorization algebra is an algebra over $\mathrm{Disj}(M)$ with values in your favorite colored operad (e.g., symmetric monoidal category).

A collection of opens $\{U_i\}$ of M is a *Weiss cover* if every finite set of points $\{x_1, \ldots, x_n\} \subset M$ is contained in some U_i. (An ordinary cover satisfies this condition for singletons.)

This notion determines a Grothendieck topology on M.

A *factorization algebra* is a prefactorization algebra that is a cosheaf for this Weiss topology.

Costello identifies a class of field theories that encompasses many interesting examples but has strict enough analytic constraints to admit explicit constructions. Basically he extracted what made the Chern-Simons & Poisson σ-model cases work.

In brief, a *classical* BV theory—in this sense—consists of

- a $\mathbb{Z}$-graded vector bundle $E \to M$ of total finite rank,
- a nondegenerate degree -1 pairing $E \otimes E \to \mathrm{Dens}_M$,
- a local action functional S whose quadratic term makes smooth sections $\mathcal{E}$ into an elliptic complex.

We require the classical master equation $\{S, S\} = 0$, where the bracket is determined by pairing.

The *classical observables* are the prefactorization algebra that assigns to $U \subset M$, the cochain complex

$$\mathrm{Obs}^{\mathrm{cl}}(U) = (\widehat{\mathrm{Sym}}(\mathcal{E}(U)^*), \{S, -\}).$$

(I should talk about functional analytic aspects, but I will suppress that.) One can use the fact that compactly supported distributions form a cosheaf to prove $\mathrm{Obs}^{\mathrm{cl}}$ is a factorization algebra.

Note that it is always a dg commutative algebra, and all the structure maps are maps of dg commutative algebras.

Costello's restrictions enable him to use rather soft techniques to do renormalization. He combines this technique with exact RG equations in the style of Polchinski to formulate a nice notion of *effective action* $\{S[\Psi]\}$ depending on a parametrix Ψ. Such an action has $\hbar$-dependent terms arising from the RG equation, which is essentially a Feynman graph expansion. We restrict to effective actions whose limit is a local functional as Ψ goes to zero.

A *quantum* BV theory is such an effective action that satisfies a quantum master equation

$$\{S[\Psi], S[\Psi]\}_\Psi + \hbar\Delta_\Psi S[\Psi] = 0,$$

where the BV bracket and Laplacian depend on the parametrix as well. A solution at one parametrix ensures a solution at every other, which is a feature of the formalism.

However, a classical theory may not admit a lift to a quantum theory, and if it does, the lift need not be unique. Such is life.

Given a BV quantization $\{S[\Psi]\}$ of a classical theory, each parametrix Ψ determines a cochain complex of global quantum observables

$$\mathrm{Obs}^{\mathrm{q}}_{\Psi}(M) = (\widehat{\mathrm{Sym}}(\mathcal{E}(M)^*)[[\hbar]], \{S[\Psi], -\}_{\Psi} + \hbar\Delta_{\Psi}),$$

but these are all isomorphic, using the RG equation. Hence we have a well-posed notion of the global observables.

Loosely speaking, one can define the *support* of a global observable as the limit of its supports as Ψ goes to zero. Then $\mathrm{Obs}^{\mathrm{q}}(U)$ is the cochain complex of observables with support in U. (Verifying this works is rather involved bookkeeping.)

This construction determines a factorization algebra Obs^{q} in dg $\mathbb{C}[[\hbar]]$-modules. Modulo $\hbar$, it is equivalent to Obs^{cl}. (It's involved bookkeeping to make sure the structure maps work. The Weiss cosheaf condition uses a simple spectral sequence argument.)

In this sense, every BV quantization determines a deformation quantization of the factorization algebra of classical observables.

A factorization algebra encodes a lot of information, and so it can be a bit unwieldy. People tend to focus on two aspects:

- the very local, such as the behavior of OPE on a small disk
- global observables on nice, closed manifolds

We have structure theorems that identify local behavior with more established types of algebra.

- We get vertex algebras from chiral CFTs in complex dimension 1.
- We get algebras over little n-disks operad from TFTs in dimension n. (Recall our examples.)

We can sometimes compute global sections by exploiting the nice analytic properties of elliptic complexes. For instance, from such chiral CFTs we recover the usual holomorphic vector bundles of conformal blocks on the moduli of curves.

- As mentioned, finding precise relations with pAQFT
- Extending to boundary conditions (CMR) & defects (e.g., Ayala-Francis)
- Understanding dualities (e.g., Costello's recent work using Koszul duality to prove a twisted form of AdS/CFT)
- Large N limits via a BV quantization of the Loday-Quillen-Tsygan theorem
- Constructing (new!) integrable hierarchies by compactification of BCOV theory (recent work of Si Li, including explanation of topological recursion)
- Of course, every important theory deserves to have its factorization algebra analyzed ...