(1) A one-relator group is a group of the form \((\mathbb{Z} * \cdots * \mathbb{Z})/N\), where

- there are \(n\) copies of \(\mathbb{Z}\), and the \(i\)th copy is generated by \(a_i\), and
- \(N\) is the smallest normal subgroup containing a fixed word \(r\) in \(a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\).

The \(a_i\) are called generators and the word \(r\) is called the relation. Given a one-relator group \(G\), use the Seifert–Van Kampen theorem to construct a path-connected space \(X\) with \(\pi_1(X) \simeq G\).

(2) More generally, to form a finitely presented group on the generators \(a_1, \ldots, a_n\) with relations \(r_1, \ldots, r_k\), one constructs \((\mathbb{Z} * \cdots * \mathbb{Z})/N\) where now \(N\) is the smallest normal subgroup of the free product containing the words \(r_1, \ldots, r_k\). Explain how to construct a space with fundamental group isomorphic to such a group. (Hint: use the fact that you can build the group \(G\) by adding one relation at a time. You need not give full details.)

(3) Use the constructions above to exhibit spaces with fundamental groups isomorphic to the following:

(a) \(\mathbb{Z}/n\mathbb{Z}\)
(b) \(\mathbb{Z}/n\mathbb{Z} \ast \mathbb{Z}/m\mathbb{Z}\)
(c) \(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}\)
(d) \(G\) a finitely generated Abelian group (any such group is isomorphic to a product of finitely many \(\mathbb{Z}\)'s and finitely many finite cyclic groups).

(4) The Borsuk–Ulam theorem says that given a continuous map \(f: S^2 \to \mathbb{R}^2\), there must exist points \(x, -x \in S^2\) such that \(f(x) = f(-x)\). Cf. Hatcher. Is the analogue of this theorem true for the torus \(S^1 \times S^1\)? In other words, given a continuous map \(f: S^1 \times S^1 \to \mathbb{R}^2\), must there exist \((x, y)\) such that \(f(x, y) = f(-x, -y)\)?