Weyl group multiple Dirichlet series

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Let $\Phi$ be an irreducible root system of rank $r$.

Our goal: explain general construction of multiple Dirichlet series in $r$ complex variables $s = (s_1, \ldots, s_r)$

$$Z(s) = \sum_{c_1, \ldots, c_r} \frac{a(c_1, \ldots, c_r)}{c_1^{s_1} \ldots c_r^{s_r}}$$

satisfying a group of functional equations isomorphic to the Weyl group $W$ of $\Phi$.

The functional equations intermix all the variables, and are closely related to the usual action of $W$ on the space containing $\Phi$. 
Example

Let $\Phi = A_2$, $W = \langle \sigma_1, \sigma_2 \mid \sigma_i^2 = 1, \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$. The desired functional equations look like

$$\sigma_1 : s_1 \rightarrow 2 - s_1, s_2 \rightarrow s_1 + s_2 - 1, \quad \sigma_2 : s_1 \rightarrow s_1 + s_2 - 1, s_2 \rightarrow 2 - s_2$$
Why?

- Such series provide tools for certain problems in analytic number theory (moments, mean values, etc.).
- Conjecturally these series arise as Fourier–Whittaker coefficients of Eisenstein series on metaplectic groups:

\[ 1 \to \mu_n \to \tilde{G}(\mathbb{A}_F) \to G(\mathbb{A}_F) \to 1 \]

This has been proved in some cases (type A and type B (double covers)).

- The series are built out of arithmetically interesting data, such as Gauss sums, \( n \)th power residue symbols, Hilbert symbols, and (sometimes) \( L \)-functions.
- The objects that arise in the construction have interesting relationships with combinatorics, representation theory, and statistical mechanics.
Let $E^*(z, s)$ be the half-integral weight Eisenstein series on $\Gamma_0(4)$:

$$E^*(z, s) = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} j_{1/2}(\gamma, z)^{-1} \Im(\gamma z)^{s/2}. $$

Maass showed that its $d$th Fourier coefficient is essentially

$$L(s, \chi_d),$$

where $\chi_d$ is the quadratic character attached to $\mathbb{Q}(\sqrt{d}/\mathbb{Q})$.

*Essentially* means up to the Euler 2-factor, archimedean factors, and certain correction factors that have to be inserted when $d$ isn’t squarefree.
Siegel, Goldfeld–Hoffstein

Siegel (1956), Goldfeld–Hoffstein (1985):

\[
Z(s, w) = \int_0^\infty (E^*(iy, s/2) - \text{const term}) y^w \frac{dy}{y}.
\]

The result is a Dirichlet series roughly of the form

\[
Z(s, w) \approx \sum_d \frac{L(s, \chi_d)}{d^w}.
\]

This behaves well in \(s\) since it’s built from the Dirichlet \(L\)-functions, and it turns out to have nice analytic properties in \(w\) as well. Goldfeld–Hoffstein used this to get estimates for sums like

\[
\sum_{|d|<X} \left| \sum_{d \text{ fund.}} L(1, \chi_d) \right|, \quad \sum_{|d|<X} \left| \sum_{d \text{ fund.}} L\left(\frac{1}{2}, \chi_d\right) \right|.
\]
$Z(s, w)$ satisfies a functional equation in $s$, again because of the Dirichlet $L$-functions. But it turns out that it satisfies extra functional equations.

In fact, $Z$ satisfies a group of 12 functional equations, and is an example of a Weyl group multiple Dirichlet series of type $A_2$. There is a subgroup of functional equations isomorphic to $S_3 = W(A_2)$, and an extra one swapping $s$ and $w$ that corresponds to the outer automorphism of the Dynkin diagram:
Connection to $A_2$

Why is this series related to root system $A_2$ (besides the fact that there are two variables and I drew the picture that way)?

Imagine expanding the $L$-functions in the rough definition:

$$Z(s, w) = \sum_{d} \frac{L(s, \chi_d)}{d^w} = \sum_{d} d^w \sum_{c} \left(\frac{d}{c}\right) c^{-s} = \sum_{d,c} \left(\frac{d}{c}\right) c^{-s} d^{-w}.$$
Heuristically, the multiple Dirichlet series looks like

\[ Z(s) = \sum_{c_1, \ldots, c_r} \frac{a(c_1, \ldots, c_r)}{c_1^{s_1} \cdots c_r^{s_r}} \]

where \( a(c_1, \ldots, c_r) \) is a product of \( n \)th power residue symbols corresponding to the edges of the Dynkin diagram.

For instance, \( D_4, n = 2 \) leads to a series related to the third moment of quadratic Dirichlet \( L \)-functions.
Setup

- $F$ number field with $2n$th roots of unity
- $S$ set of places of $F$ containing archimedean, ramified, and such that $\mathcal{O}_S$ is a PID
- $\Phi$ irreducible simply-laced root system of rank $r$
- $\{\alpha_1, \ldots, \alpha_r\}$ the simple roots
- $m = (m_1, \ldots, m_r)$ $r$-tuple of integers in $\mathcal{O}_S$
- $s = (s_1, \ldots, s_r)$ $r$-tuple of complex variables
Setup

- $F_S = \prod_{v \in S} F_v$
- $\mathcal{M}(\Phi)$ certain finite-dimensional space of complex-valued functions on $(F_S^\times)^r$ (to deal with Hilbert symbols and units)
- $\Psi \in \mathcal{M}(\Phi)$
- $H(c; m)$ to be specified later ... this is the most important part of the definition
Then the multiple Dirichlet series looks like

\[ Z(s; m, \Psi; \Phi, n) = \sum_c \frac{H(c; m) \Psi(c)}{\prod |c_i|^{s_i}} , \]

where \( c = (c_1, \ldots, c_r) \) and each \( c_i \) ranges over \( (\mathcal{O}_S \setminus \{0\})/\mathcal{O}_S^\times \).
The function $H$

The coefficients $H$ have to be carefully defined to guarantee that $Z$ satisfies the desired group of functional equations. General considerations tell us how to define $H$ in the following cases:

- When $c_1 \cdots c_r$ and $c'_1 \cdots c'_r$ are relatively prime, one uses a "twisted multiplicativity" to construct $H(cc'; m)$ from $H(c; m)$ and $H(c'; m)$. One puts

$$H(cc'; m) = \varepsilon(c, c') H(c; m) H(c'; m),$$

where $\varepsilon(c, c')$ is a root of unity built out of residue symbols and root data:

$$\varepsilon(c, c') = \prod_{i=1}^{r} \left( \frac{c_i}{c'_i} \right) \left( \frac{c'_i}{c_i} \right) \prod_{i < j} \left( \frac{c_i}{c'_j} \right) \left( \frac{c'_i}{c_j} \right).$$
The function $H$

When $(c_1 \cdots c_r, m'_1 \cdots m'_r) = 1$, we can define $H(c; mm')$ in terms of $H(c; m)$ and certain power residue symbols:

$$H(c; mm') = \prod_{j=1}^{r} \left( \frac{m'_j}{c_j} \right) H(c; m)$$
The function $H$

So we reduce the definition of $H$ to that of

$$H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}),$$

where $\varpi$ is a prime in $\mathcal{O}_S$.

This leads naturally to the generating function

$$N = N(x_1, \ldots, x_r)$$

$$= \sum_{k_1, \ldots, k_r \geq 0} H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}) x_1^{k_1} \cdots x_r^{k_r}$$

($m$ is fixed). One can ask what properties this series has to satisfy so that one can prove $Z$ satisfies the right group of functional equations.
The function $N$

\[ N = N(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 0} H(\omega^{k_1}, \ldots, \omega^{k_r}) x_1^{k_1} \cdots x_r^{k_r}. \]

If one puts $x_i = q^{-s_i}$, where $q = |\mathcal{O}_S/\mathfrak{o}|$, then one can see that the global functional equations imply $N$ must transform a certain way under a certain $W$-action.

This leads to a connection with characters of representations of $\mathfrak{g}$, the simple complex Lie algebra attached to $\Phi$.

In this relationship the monomials correspond to certain weight spaces.
Building $N$

The connection with characters leads to two approaches to defining $N$:

- **Crystal graphs.** These are models for $\mathfrak{g}$ representations that have various combinatorial incarnations (Gelfand–Tsetlin patterns, tableaux, Proctor patterns, Littlemann path model, ...). One tries to extract a statistic from the combinatorial model to define the coefficients of $N$. (Brubaker–Bump–Friedberg, Beineke–Brubaker–Frechette, Chinta–PG)

- **Weyl character formula.** This is an explicit expression for a given character as a ratio of two polynomials. We take this approach and define a deformation of Weyl’s formula that reflects the metaplectacticy (metaplectaciousness?) of the setup. (Chinta–PG, Bucur–Diaconu)
\begin{itemize}
  \item $\Lambda_w$ weight lattice of $\Phi$
  \item $\{\omega_1, \ldots, \omega_r\}$ fundamental weights
  \item $\rho = \sum \omega_i$ the Weyl vector
  \item $\mathbb{Z}[y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$ group ring of the weight lattice ($y_i \leftrightarrow \omega_i$)
  \item $\theta$ a dominant weight
\end{itemize}

Then according to Weyl the character of the irreducible representation of highest weight $\theta$ is

$$
\chi_{\theta} = \frac{\sum_{w \in W} \text{sgn}(w) y^{w(\theta + \rho) - \rho}}{\prod_{\alpha > 0} (1 - y^{-\alpha})} = \sum_{w \in W} \text{sgn}(w) y^{w(\theta + \rho) - \rho} \frac{1}{\Delta(y)}.
$$

$$
\Delta(y) = \prod_{\alpha > 0} (1 - y^{-\alpha}).
$$
Our goal now is to define the $W$-action leading to $H$. For the application to multiple Dirichlet series, we normalize things slightly differently. Thus we work with the root lattice, introduce some $q = |\mathcal{O}_S/\varpi|$ powers, shift the character around, …

- $\Lambda$ root lattice of $\Phi$
- $d: \Lambda \to \mathbb{Z}$ height function on the roots
- $A \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ complex group ring of $\Lambda$ ($x_i \leftrightarrow \alpha_i$)
- $\tilde{A} \simeq \mathbb{C}(x_1, \ldots, x_r)$ fraction field of $A$
- $\theta = \rho + \sum l_i \omega_i$ a strictly dominant weight (recall that we’re defining $H(c; m)$ when $m = (\varpi^{l_1}, \ldots, \varpi^{l_r})$)
The action on monomials

We let the Weyl group act on monomials through a “change of variables” map. This is essentially the same as the geometric action of $W$ on the root lattice (except for the $q$ power).

If $f(x) = x^\beta$, we put

$$f(wx^\beta) = q^{d(w^{-1}\beta - \beta)} x^{w^{-1}\beta}.$$
Affine action of $W$

Given any $\lambda \in \Lambda$, we put

$$w \cdot \lambda = w(\lambda - \theta) + \theta,$$

where the action on the right hand side is the usual action on the root lattice. This just performs an affine reflection of $\Lambda \otimes \mathbb{R}$ (the same as the usual $w$ reflection but shifted to have center $\theta$).

If $\sigma_i$ is a simple reflection, we put

$$\mu_i(\lambda) = d(\sigma_i \cdot \lambda - \lambda).$$

This is just the multiple of $\alpha_i$ needed to go from $\lambda$ to $\sigma_i \cdot \lambda$. 
Affine action of $W$

\[ \theta \]

\[ \lambda \]

\[ \sigma_1 \bullet \lambda \]
Gauss sums

Choose some complex numbers $\gamma(i)$, $i = 1, \ldots, n - 1$ such that $\gamma(i)\gamma(n - i) = 1/q$. Put $\gamma(0) = -1$.

Ultimately these numbers will be Gauss sums (the same ones appearing in the metaplectic cocycle), but actually any complex numbers satisfying these relations will work.

Extend $\gamma(i)$ to all integers by reducing $i \mod n$. 
Homogeneous decomposition

The action on a monomial $f(x) = x^\beta$ depends on the congruence class of the monomial mod $n\Lambda$.

To treat general rational functions, we decompose $\tilde{A}$ into homogeneous parts

$$\tilde{A} = \bigoplus_{\lambda \in \Lambda/n\Lambda} \tilde{A}_\lambda.$$ 

$A_\lambda$ consists of those rational function $f/g$ where all monomials in $g$ lie in $n\Lambda$ and those in $f$ map to $\lambda$ modulo $n\Lambda$.

e.g.,

$$\frac{1 - xy}{x^2 - y^2} = \frac{1}{x^2 - y^2} - \frac{xy}{x^2 - y^2}$$
Finally

**Theorem** (Chinta–PG) Suppose $f \in A_\beta$. Let $\sigma_i$ be a simple reflection and let $(k)_n$ be the remainder upon division of $k$ by $n$. Then

$$
(f|_{\theta \sigma_i})(x) = (qx_i)^{l_i+1-(\mu_i(\beta))_n} \frac{1 - 1/q}{1 - q^{n-1}x_i^n} f(\sigma_i x) 
(P)
$$

$$
- \gamma(\mu_i(\beta)) \cdot (qx_i)^{l_i+1-n} \frac{1 - (qx_i)^n}{(1 - q^{n-1}x_i^n)} f(\sigma_i x) 
(Q)
$$

extends to a $W$-action on $\mathbb{C}(x_1, \ldots, x_r)$. 

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The $W$-action
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$\mathbf{x}^\beta$
The $W$-action

\[ x^\beta \]

\[ \sigma_i \cdot x^\beta \]
The $W$-action

\[ x^\beta \]

\[ Q \]

\[ \sigma_i \cdot x^\beta \]

\[ P \]
Theorem (Chinta–PG)

- Put $\Delta(x) = \prod_{\alpha > 0} (1 - q^n x^{n\alpha})$ and $D(x) = \prod_{\alpha > 0} (1 - q^{n-1} x^{n\alpha})$.
  Then
  \[
  h(x) = \sum_{w \in W} \frac{(1|\theta w)(x)}{\Delta(wx)}
  \]
  is a rational function such that $hD$ is a polynomial.

- Let $N = hD$, define $H$ by
  \[
  N = \sum_{k_1, \ldots, k_r \geq 0} H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}) x_1^{k_1} \cdots x_r^{k_r},
  \]
  and specialize the $\gamma(i)$ to the appropriate Gauss sums. Then the resulting multiple Dirichlet series $Z(s; m, \Psi; \Phi, n)$ has analytic continuation to $\mathbb{C}^r$ and satisfies a group of functional equations isomorphic to $W$.
Here $g_1 = q \gamma(1)$ and the notation $(a, b)$ means

$$\theta = (a + 1) \omega_1 + (b + 1) \omega_2.$$

- $(0, 0)$: $1 + g_1 x + g_1 y - g_1 q x^2 y - g_1 q x y^2 - q^2 x^2 y^2$
- $(1, 0)$: $1 - q x^2 + g_1 y - g_1 q x^2 y + g_1 q^2 x^2 y + q^3 x^3 y - g_1 q^3 x^2 y^3 - q^4 x^3 y^3$
- $(1, 1)$: $1 - q x^2 - q y^2 + q^2 x^2 y^2 - q^3 x^2 y^2 + q^4 x^4 y^2 + q^4 x^2 y^4 - q^5 x^4 y^4$
- $(2, 1)$:
  
  $1 - q x^2 + q^2 x^2 + g_1 q^2 x^3 - q y^2 + q^2 x^2 y^2 - 2 q^3 x^2 y^2 + q^4 x^2 y^2 - g_1 q^3 x^3 y^2 + g_1 q^4 x^3 y^2 + q^4 x^4 y^2 - q^5 x^4 y^2 - g_1 q^5 x^5 y^2 + q^4 x^2 y^4 - q^5 x^2 y^4 - g_1 q^5 x^3 y^4 + g_1 q^6 x^3 y^4 - q^5 x^4 y^4 + q^6 x^4 y^4 + g_1 q^6 x^5 y^4 - g_1 q^7 x^5 y^4 + q^7 x^3 y^5 - q^8 x^5 y^5$
Open questions

- The WCF method works for all $\Phi$, whereas the crystal graph approach has only been worked out for some (classical) $\Phi$. Can one do the latter for all $\Phi$ uniformly? (Kim–Lee, McNamara)

- Prove that $Z$ is a Whittaker coefficient of a metaplectic Eisenstein series. (Chinta–Offen)

- Prove that the crystal graph descriptions and the WCF descriptions coincide. (Chinta–Offen + McNamara)

- Develop multiple Dirichlet series on affine Weyl groups and crystallographic Coxeter groups (Bucur–Diaconu, Lee)

- What is the geometric interpretation of Weyl group multiple Dirichlet series over function fields?
References


