ON THE TOPOLOGICAL COMPUTATION OF $K_4$ OF THE GAUSSIAN AND EISENSTEIN INTEGERS

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Abstract. In this paper we use topological tools to investigate the structure of the algebraic $K$-groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$, where $i := \sqrt{-1}$ and $\rho := (1 + \sqrt{-3})/2$. We exploit the close connection between homology groups of $GL_n(\mathbb{R})$ for $n \leq 5$ and those of related classifying spaces, then compute the former using Voronoi’s reduction theory of positive definite quadratic and Hermitian forms to produce a very large finite cell complex on which $GL_n(\mathbb{R})$ acts. Our main result is that $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ have no $p$-torsion for $p \geq 5$.

1. Introduction

1.1. Statement of results. Let $R$ be the ring of integers of a number field $F$. Only very few cases are known where the algebraic $K$-group $K_4(R)$ has been explicitly computed, the first such $K_4(\mathbb{Z})$ having been determined as recently as 2000 by Rognes [17], building on work of Soulé [18]. The goal of this paper is the explicit topological computation of the torsion (away from 2 and 3) in the groups $K_4(R)$ for $R$ one of two special imaginary quadratic examples: the Gaussian integers $\mathbb{Z}[i]$ and the Eisenstein integers $\mathbb{Z}[\rho]$, where $i := \sqrt{-1}$ and $\rho := (1 + \sqrt{-3})/2$. Our work is in the spirit of Lee–Szczarba [12–14], Soulé [19], and Elbaz-Vincent–Gangl–Soulé [7, 8] who treated $K_N(\mathbb{Z})$ for small $N$, and Staffeldt [20] who investigated $K_3(\mathbb{Z}[i])$. As in these works, the first step is to compute the cohomology of $GL_n(\mathbb{R})$ for $n \leq N + 1$; information from this computation is then assembled into information about the $K$-groups following the program in §1.2. Using these computations we show the following (Theorem 4.1):

Theorem 1.1. The orders of the groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ are not divisible by any primes $p \geq 5$.

We remark that this result is not new; in fact, Kolster’s work [11] implies the stronger result that $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ vanish. Indeed, if $R$ is the ring of integers of a CM field, then Kolster proved that, assuming the Quillen–Lichtenbaum conjecture, the orders of the groups $K_4(R)$, $n = 1, 2, 3, \ldots$, can be computed in terms of special values of certain $L$-functions. This deep connection between $K$-groups and special values of $L$-functions is now a theorem, thanks to the celebrated work by Voevodsky [21] and Rost, as put into context in [9].

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Our work, on the other hand, treats $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ by completely different methods. We only use the definition of the $K$-groups and explicit results about the cohomology of the relevant arithmetic groups \cite{6}, together with Arlettaz’s bounds on the kernel of the Hurewicz homomorphism \cite{1}, to prove Theorem 4.1. This also explains why our calculations do not allow us to say anything for the primes 2 and 3: both the results of \cite{6} and the injectivity of the Hurewicz map in our cases only hold away from these primes.

1.2. Outline of method. In the rest of this introduction we outline the main steps of our argument. These follow the classical approach for computing algebraic $K$-groups of number rings due to Quillen \cite{15}, which shifts the focus to computing the homology (with nontrivial coefficients) of certain arithmetic groups.

(i) (Definition) By definition the algebraic $K$-group $K_N(R)$ of a ring $R$ is a particular homotopy group of a topological space associated to $R$: we have $K_N(R) = \pi_{N+1}(BQ(R))$, where $BQ(R)$ is a certain classifying space attached to the infinite general linear group $GL(R)$. In particular $BQ(R)$ is the classifying space of the category $Q(R)$ of finitely generated $R$-modules. This is known as Quillen’s $Q$-construction of algebraic $K$-theory \cite{16}.

(ii) (Homotopy to Homology) The Hurewicz homomorphism $\pi_{N+1}(BQ(R)) \to H_{N+1}(BQ(R))$ allows one to replace the homotopy group by a homology group without losing too much information; more precisely, what may get lost is information about small torsion primes appearing in its finite kernel.

(iii) (Stability) By a stability result of Quillen \cite{15} p. 198 one can pass from $Q(R)$ to the category $Q_{N+1}(R)$ of finitely generated $R$-modules of rank $\leq N$ for sufficiently large $N$. This amounts to passing from $GL(R)$ to the finite-dimensional general linear group $GL_{N+1}(R)$.

(iv) (Sandwiching) The homology groups to be determined are then $H_n(BQ_n(R))$ for $n \leq N + 1$. Rather than compute these directly, one uses the fact that they can be sandwiched between homology groups of $GL_n(R)$, where the homology is taken with (nontrivial) coefficients in the Steinberg module $St_n$ associated to $GL_n(R)$.

(v) (Voronoi homology) The standard method to compute the homology groups $H_m(GL_n(R), St_n)$ for a number ring $R$ is via Voronoi complexes. These are the chain complexes of certain explicit polyhedral reduction domains of a space of positive definite quadratic or Hermitian forms of a given rank, depending respectively on whether $R = \mathbb{Z}$ or $R$ is imaginary quadratic. The Voronoi complex provides most of the desired information on the homology in question: as in (iv), one might again lose information about small primes—in particular, such information could be hidden in the higher differentials of a spectral sequence involving the stabilizers of cells in the Voronoi complex. In any case, one can usually find a small upper bound on the sizes of those primes, which means that one can effectively determine the homology and ultimately the $K$-groups modulo small primes.

(vi) (Vanishing Results) There are various techniques to show vanishing of homology groups. As a starting point one has vanishing results for $H_m(BQ_1)$ as in Theorem \cite{5} below, and for $H_0(GL_n, St_n)$ as in Lee–Szczarba \cite{14}. 

For a given N, using (ii) and knowing the results of (iv)–(vi) for all 0 ≤ n ≤ N + 1 is often enough to give a bound p ≤ B on the primes p dividing the order of the torsion subgroup \( K_{N,\text{tors}}(R) \) of \( K_N(R) \).

1.3. Outline of paper. In this paper the sections work backwards through the method outlined in §1.2 to determine the structure of \( K_4(\mathbb{Z}[i]) \) and \( K_4(\mathbb{Z}[\rho]) \). In §2 we describe the computation of the Voronoi homology of these two number rings (i.e., step (v) above). In §3 we use the Voronoi homology and some vanishing results to determine the groups \( H_m(BQ_n(R)) \) (i.e., step (iv) above). A key role here is played by Quillen’s stability result (iii) for \( BQ_n \), which serves as a stopping criterion. Finally, in §4 we work out the potential primes entering the kernel of the Hurewicz homomorphism (i.e., step (ii) above), which gives Theorem 1.1.

2. Homology of Voronoi complexes

We first collect the results from [6] concerning the Voronoi complexes attached to \( \Gamma = \text{GL}_m(\mathbb{Z}[i]) \) or \( \Gamma = \text{GL}_m(\mathbb{Z}[\rho]) \); this is the necessary information needed for step (v) from §1.2 above. More details about these computations, including background about how the computations are performed, can be found in [6].

Let \( F \) be an imaginary quadratic field with ring of integers \( R \), and let \( X_n := \text{GL}_n(\mathbb{C})/U(n) \) be the symmetric space of \( \text{GL}_n(F \otimes \mathbb{R}) \). The space \( X_n \) can be realized as the quotient of the cone of rank \( n \) positive definite Hermitian matrices \( \mathcal{H}_n \) modulo homotheties (i.e. non-zero scalar multiplication), and a partial Satake compactification \( X^\ast_n \) of \( X_n \) is given by adjoining boundary components to \( X_n \) given by the cones of positive semi-definite Hermitian forms with an \( F \)-rational nullspace (again taken up to homotheties). We let \( \partial X^\ast_n := X^\ast_n \backslash X_n \) denote the boundary of \( X^\ast_n \). Then \( \Gamma := \text{GL}_n(R) \) acts by left multiplication on both \( X_n \) and \( X^\ast_n \), and the quotient \( \Gamma \backslash X^\ast_n \) is a compact Hausdorff space.

A generalization—due to Ash [2, Chapter II] and Koecher [10]—of the polyhedral reduction theory of Voronoi [22] yields a \( \Gamma \)-equivariant explicit decomposition of \( X^\ast_n \) into (Voronoi) cells. Moreover, there are only finitely many cells modulo \( \Gamma \). Let \( \Sigma_d := \Sigma_d(\Gamma) \) be a set of representatives of the \( \Gamma \)-inequivalent \( d \)-dimensional Voronoi cells that meet the interior \( X_n \), and let \( \Sigma_d := \Sigma_d(\Gamma) \) be the subset of representatives of the \( \Gamma \)-inequivalent orientable cells in this dimension; here we call a cell orientable if all the elements in its stabilizer group preserve its orientation. One can form a chain complex \( \text{Vor}_n \), the Voronoi complex, and one can prove that modulo small primes the homology of this complex is the homology \( H_*(\Gamma, S^n) \), where \( S^n \) is the rank \( n \) Steinberg module (cf. [4] p. 437). To keep track of these small primes explicitly, we make the following definition.

Definition 2.1 (Serre class of small prime power groups). Given \( k \in \mathbb{N} \), we let \( S_{p,k} \) denote the Serre class of finite abelian groups \( G \) whose cardinality \( |G| \) has all of its prime divisors \( p \) satisfying \( p \leq k \).

For any finitely generated abelian group \( G \), there is a unique maximal subgroup \( G_{p,k} \) of \( G \) in the Serre class \( S_{p,k} \). We say that two finitely generated abelian groups \( G \) and \( G' \) are equivalent modulo \( S_{p,k} \) and write \( G \cong_{p,k} G' \) if the quotients \( G/G_{p,k} \cong G'/G'_{p,k} \) are isomorphic.

Theorem 2.2 ([6, Theorem 3.7]). Let \( b \) be an upper bound on the torsion primes for \( \text{GL}_n(R) \). Then \( H_m(\text{Vor}_n) \cong_{p,k} H_{m-n+1}(\text{GL}_n(R), S^n) \).
2.1. Voronoi data for $R = \mathbb{Z}[i]$. We now give results for the Voronoi complexes and their homology in the cases relevant to our paper. This subsection treats the Gaussian integers; in §2.2 we treat the Eisenstein integers.

**Theorem 2.3 ([20]).**

1. There is one $d$-dimensional Voronoi cell for $GL_2(\mathbb{Z}[i])$ for each $1 \leq d \leq 3$, and only the 3-dimensional cell is orientable.

2. The number of $d$-dimensional Voronoi cells for $GL_3(\mathbb{Z}[i])$ is given by:

$$
\begin{array}{cccccccc}
\text{d} & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\Sigma_d(GL_2(\mathbb{Z}[i])) & 2 & 3 & 4 & 5 & 3 & 1 & 1 \\
\Sigma_d(GL_3(\mathbb{Z}[i])) & 0 & 0 & 1 & 4 & 3 & 0 & 1 \\
\end{array}
$$

**Theorem 2.4 ([6, Table 12]).** The number of $d$-dimensional Voronoi cells for $GL_4(\mathbb{Z}[i])$ is given by:

$$
\begin{array}{cccccccccc}
\text{d} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\Sigma_d(GL_4(\mathbb{Z}[i])) & 4 & 10 & 33 & 98 & 258 & 501 & 704 & 628 & 369 & 130 & 7 & 2 \\
\Sigma_d(GL_4(\mathbb{Z}[i])) & 0 & 0 & 5 & 48 & 189 & 435 & 639 & 597 & 346 & 120 & 22 & 2 & 2 \\
\end{array}
$$

We remark that for $GL_3(\mathbb{Z}[i])$ the Voronoi complexes and their homology ranks were originally computed by Staﬀeldt [20], who even distilled the 3-part for each homology group. After calculating the diﬀerentials for this complex one obtains the following homology groups, in agreement with Staﬀeldt’s results:

**Theorem 2.5 ([20, Theorems IV, 1.3 and 1.4, p.785]).**

(1) $H_m(GL_2(\mathbb{Z}[i]), St_2) \cong_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise,} \end{cases}$

(2) $H_m(GL_3(\mathbb{Z}[i]), St_3) \cong_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise.} \end{cases}$

In particular, from the above theorem we deduce that the only possible torsion primes for $H_m(GL_n(\mathbb{Z}[i]), St_n)$ for $n = 2, 3$ are the primes 2 and 3.

For $GL_4(\mathbb{Z}[i])$, the last column of [6 Table 12] shows that the elementary divisors of all the diﬀerentials in the Voronoi complex are supported on primes $\leq 5$. In fact a closer examination of this table reveals the following:

**Theorem 2.6 ([6 Theorem 7.2 and Table 12]).**

(3) $H_m(GL_4(\mathbb{Z}[i]), St_4) \cong_{/p \leq 5} \begin{cases} \mathbb{Z}^2 & \text{if } m = 5, \\ \mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\ 0 & \text{otherwise.} \end{cases}$

Moreover, the only degrees where 5-torsion could occur are $m = 1, 6$ or $m \geq 10$.

From this we see that there is the potential for 5-torsion for $H_m(GL_4(\mathbb{Z}[i]), St_4)$. While there is 5-torsion in $H_{10}$, and possibly further 5-torsion in $H_m$ for $m \geq 6$, we will show that for degree $m = 1$ (the only relevant degree for the $K$-groups we consider) the group $H_1$ contains no 5-torsion (Proposition 2.7).
In order to analyze $H_1(\text{GL}_4(\mathbb{Z}[i]), St_4)$ more closely, we will need to use spectral sequences. According to [5, VII.7] there is a spectral sequence $E_{d,q}^1$ converging to the equivariant homology groups $H^H_{d+q}(X_\sigma, \partial X_\sigma; \mathbb{Z})$ of the homology pair $(X_\sigma, \partial X_\sigma)$, and such that

$$E_{d,q}^1 = \bigoplus_{\sigma \in \Sigma^*_d} H_q(\Gamma_\sigma, \mathbb{Z}_{\sigma}),$$

where $\mathbb{Z}_{\sigma}$ is the orientation module of the cell $\sigma$.

**Theorem 2.8.** The group $H_1(\text{GL}_4(\mathbb{Z}[i]), St_4) \cong \bigcap_{p \leq 3} \{0\}$.

**Proof.** Since $H_1(\text{GL}_4(\mathbb{Z}[i]), St_4)$ is a subquotient of $\bigoplus_{d+q=4} E_{d,q}^1$, we consider the individual summands $E_{d,4-d}^1$ for $0 \leq p \leq 4$:

- Since there are no cells in $\Sigma^*_d$ for $d \leq 2$, we have $E_{0,4}^1 = E_{1,3}^1 = E_{2,2}^1 = 0$.
- Consider now $d = 3$. There are four cells in $\Sigma^*_3$, and for each of them the index 2 subgroup acting trivially on the orientation module has an abelian-ization $\mathbb{Z}/2\mathbb{Z}$ up to 2-groups. Thus in particular we have

$$E_{3,1}^1 = \bigoplus_{\sigma \in \Sigma^*_3} H_1(\text{Stab}(\sigma), \mathbb{Z}_{\sigma}) \in S_{p \leq 3},$$

where $S_{p \leq 3}$ is as in Definition 2.1, and therefore this term contains no 5-torsion.

- Finally, for $d = 4$, there is only one cell (out of ten) in $\Sigma^*_4$, denoted by $\sigma^*_4$, that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in $H_1(\text{Stab}(\sigma^*_4), \mathbb{Z})$ (where $\mathbb{Z}$ is the orientation module $\mathbb{Z}_{\sigma}$). Indeed, the order-preserving subgroup $K_1$ of $\text{Stab}(\sigma^*_4)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times A_5$, where $A_5$ is the alternating group on five letters, with abelian-ization $H_1(\text{Stab}(\sigma^*_4), \mathbb{Z}) = H_1(K_1, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$, which lies in $S_{p \leq 3}$. Thus there can be no 5-torsion from here, which completes the proof.

$\square$

2.2. **Voronoi homology data for $R = \mathbb{Z}[\rho]$.** Now we turn to the Eisenstein case.

**Theorem 2.8 (\cite{6}, Tables 1 and 11).**

1. There is one $d$-dimensional Voronoi cell for $\text{GL}_2(\mathbb{Z}[\rho])$ for each $1 \leq d \leq 3$, and only the 3-dimensional cell is orientable.

2. The number of $d$-dimensional Voronoi cells for $\text{GL}_3(\mathbb{Z}[\rho])$ is given by:

<table>
<thead>
<tr>
<th>$d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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<tbody>
<tr>
<td>$</td>
<td>\Sigma_d(\text{GL}_3(\mathbb{Z}[\rho]))</td>
<td>^1$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
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<tr>
<td>$</td>
<td>\Sigma_d(\text{GL}_3(\mathbb{Z}[\rho]))</td>
<td>$</td>
<td>0</td>
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<td>1</td>
<td>2</td>
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</tbody>
</table>

3. The number of $d$-dimensional Voronoi cells for $\text{GL}_4(\mathbb{Z}[\rho])$ is given by:

<table>
<thead>
<tr>
<th>$d$</th>
<th>3</th>
<th>4</th>
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<tr>
<td>$</td>
<td>\Sigma_d(\text{GL}_4(\mathbb{Z}[\rho]))</td>
<td>^1$</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>34</td>
<td>82</td>
<td>166</td>
<td>277</td>
<td>324</td>
<td>259</td>
<td>142</td>
<td>48</td>
</tr>
<tr>
<td>$</td>
<td>\Sigma_d(\text{GL}_4(\mathbb{Z}[\rho]))</td>
<td>$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>50</td>
<td>129</td>
<td>228</td>
<td>286</td>
<td>237</td>
<td>122</td>
<td>36</td>
</tr>
</tbody>
</table>

After calculating the differentials we find the same results as for the homology of $\mathbb{Z}[i]$ above:
Theorem 2.9 (\cite{6} Theorems 7.1 and 7.2 with Propositions 3.2 and 3.6).

(4) \[ H_m(\text{GL}_2(\mathbb{Z}[\rho]), St_2) \cong_{|p| < 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise}, \end{cases} \]

(5) \[ H_m(\text{GL}_3(\mathbb{Z}[\rho]), St_3) \cong_{|p| < 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise}, \end{cases} \]

(6) \[ H_m(\text{GL}_4(\mathbb{Z}[\rho]), St_4) \cong_{|p| < 3} \begin{cases} \mathbb{Z}^2 & \text{if } m = 5, \\ \mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\ 0 & \text{otherwise}. \end{cases} \]

Proof. Since the ranks of the homology groups in question have been computed in \cite{6}, we only have to consider the torsion in the respective groups. For fixed $n$, any torsion prime of the homology groups $H_m(\text{GL}_n(\mathbb{Z}[\rho]), St_n)$ must either divide the order of the stabilizer of some cell in $\Sigma_d^*$ for appropriate $d$, or must divide an elementary divisor of the differentials in the corresponding Voronoi complex. We consider these two possibilities in turn.

First we consider the stabilizers. For ranks $n = 2, 3$, all stabilizers of cells in $\Sigma_d^*$ lie in $S_{p<3}$. For rank $n = 4$, the prime $p = 5$ is the only torsion prime $> 3$ occurring for stabilizer orders in $\Sigma_d^*$, more precisely it occurs for $d = 9$ (two cells), $d = 14$ (two cells) and $d = 15$ (one cell).

Next we consider elementary divisors. In rank $n = 2$, the elementary divisors occurring are all even, and apart from $m = 2$, where $H_2(\text{GL}_2(\mathbb{Z}[\rho]), St_2) = H_3(\text{Vor}_r) = \mathbb{Z}$ modulo $S_{p<3}$, we have $H_m(\text{GL}_2(\mathbb{Z}[\rho]), St_2) = H_{m+1}(\text{Vor}_r) = 0$ modulo $S_{p<3}$. In rank $n = 3$, the only non-trivial elementary divisor for any of the differentials involved is $9$, arising from $d_8^1 : E_{8,0}^1 \rightarrow E_{8,0}^1$. Moreover, we get $H_m(\text{GL}_3(\mathbb{Z}[\rho]), St_3) = H_{m+2}(\text{Vor}_r) = \mathbb{Z}$ modulo $S_{p<3}$ for $m = 2, 3$ or $6$, and is zero otherwise. Finally, for rank $n = 4$, the only torsion prime $> 3$ for the homology groups $H_{m+3}(\text{Vor}_r)$ is $d = 5$, which divides the elementary divisor $15$ of $d_{14}^1$. This completes the proof. \qed

As with $\mathbb{Z}[i]$, a more refined analysis of the $\text{GL}_4(\mathbb{Z}[\rho])$ case shows that $H_1$ contains no 5-torsion:

Proposition 2.10.

(7) \[ H_1(\text{GL}_4(\mathbb{Z}[\rho]), St_4) \cong_{|p| < 3} \{0\}. \]

Proof. The argument is very similar to that of the proof of Proposition 2.7. In rank 4, we have that $H_1(\text{GL}_4(\mathbb{Z}[\rho]), St_4)$ is a subquotient of

(8) \[ \bigoplus_{d+q=4} E_{d,q}^1 = \bigoplus_{d+q=4} H_d(\Gamma_\sigma, \mathbb{Z}_\sigma). \]

We consider each of these summands in turn.

If $d \leq 2$, then there are no cells of dimension $d$ to worry about. For $d = 3$, there are two cells in $\Sigma_3^*$, with stabilizer in $S_{p<3}$, and hence

\[ E_{3,1}^1 = \bigoplus_{\sigma \in \Sigma_3^*} H_1(\text{Stab}(\sigma), \mathbb{Z}_\sigma) \not\in S_{p<3}. \]
Finally suppose $d = 4$. Then $E^{1}_{4,0} = 0 \mod S_{p<2}$, as none of the 5 classes in dim 4 is orientable. Thus modulo $S_{p<3}$ all summands in (8) vanish, which completes the proof. □

3. Vanishing and sandwiching

In this section, we carry out the sandwiching argument (step (iv) of §1.2). As a first step we invoke a vanishing result for homology groups for $BQ_1$ due to Quillen [15, p.212]. In our cases this result boils down to the following statement:

**Theorem 3.1.** For the rings $R = \mathbb{Z}[i]$ and $\mathbb{Z}[\rho]$, we have

$$H_n(BQ_1) = 0 \quad \text{whenever } n \geq 3.$$ 

For $R = \mathbb{Z}[i]$ a slightly stronger result is proved in [20] Lemma I.1.2. However, we will not need this stronger result for $\mathbb{Z}[i]$, or its analogue for $\mathbb{Z}[\rho]$.

Using our homology data from §2 and Theorem 3.1 we can get for both rings $R = \mathbb{Z}[i]$ and $R = \mathbb{Z}[\rho]$ the following result:

**Proposition 3.2.** $H_5(BQ) \cong_{/p<3} \mathbb{Z}.$

**Proof.** We will successively determine $H_5(BQ_j)$ for $j = 1, \ldots, 5$ and then identify the last group via stability with $H_5(BQ)$. For this, we will combine results from §2 with Quillen’s long exact sequence for different $r$, given by

$$\cdots \to H_n(BQ_{r-1}) \to H_n(BQ_r) \to H_n(\text{GL}_r, St_r) \to H_{n-1}(BQ_{r-1}) \to \cdots.$$ 

The case $j = 1$. By Theorem 3.1 we have $H_n(BQ_1) = 0$ for $n \geq 3$.

The case $j = 2$. From the above sequence (9) for $r = 2$, we get

$$H_5(BQ_1) \to H_5(BQ_2) \to H_3(\text{GL}_2, St_2) \to H_4(BQ_1),$$

whence $H_5(BQ_2) = 0 \mod S_{p<3}$ by (1) and (4).

The case $j = 3$. Now we invoke another result of Staffeldt’s who showed (see [20] proof of Theorem I.1.1) that

$$H_4(BQ_2) = H_4(BQ_3) = \mathbb{Z} \mod S_{p<3}. \tag{10}$$

From (9) for $r = 3$ we get the exact sequence, working mod $S_{p<3}$,

$$H_5(BQ_2) \to H_5(BQ_3) \to H_2(\text{GL}_3, St_3) \to H_4(BQ_2) \to H_4(BQ_3) \to H_1(\text{GL}_3, St_3),$$

$$= \mathbb{Z} \text{ (by (10))} \quad = \mathbb{Z} \text{ (by (10))} \quad = \mathbb{Z} \text{ (by (10))} \quad = 0 \text{ (by (9)).}$$

Since the leftmost group $H_5(BQ_2)$ vanishes modulo $S_{p<3}$ by the case $j = 2$, this sequence implies that $H_5(BQ_3) = \mathbb{Z} \mod S_{p<3}$.

The case $j = 4$. Moreover, since $H_2(\text{GL}_4, St_4) = H_1(\text{GL}_4, St_4) = 0 \mod S_{p<3}$ by Theorem 2.6 and Proposition 2.7 the sequence (9) for $r = 4$ gives in a similar way that

$$H_5(BQ_4) = H_5(BQ_5) = \mathbb{Z} \mod S_{p<3}. \tag{11}$$

The case $j = 5$. This is the most complicated of all the cases to handle. Note that $BQ$ is an $H$-space which implies that $H_1(BQ) \otimes \mathbb{Q}$ is the enveloping algebra of $\pi_{*}(BQ) \otimes \mathbb{Q}$. We know that $K_0(\mathbb{Z}[i]) = \mathbb{Z}$, $K_1(\mathbb{Z}[i]) = \mathbb{Z}/2$ and $K_2(\mathbb{Z}[i]) = 0$. 

invoking Quillen’s result that $K$ of the corresponding Hurewicz homomorphism $K$ space. Hence a theorem due to Arlettaz [1, Theorem 1.5] shows that the kernel or $Z$ denoted certainly annihilated by 144 (cf. Definition 1.3 in loc.cit., where this number is $p$ cannot contain any $S$ from $H$ and using the above result that $H$ of $H$ and $\rho$ [3, Appendix] as well as $8$ M. D. Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, On the cohomology of linear groups over imaginary quadratic fields, J. Pure Appl. Algebra 220 (2016), no. 7, 2564–2589.

The groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[p])$ lie in $S_{p<3}$.

References


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