ON $K_4$ OF THE GAUSSIAN AND EISENSTEIN INTEGERS

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Abstract. Let $R$ be either the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ or the Eisenstein integers $\mathbb{Z}[(1 + \sqrt{-3})/2]$. In this paper we investigate the algebraic $K$-group $K_4(R)$. The main method exploits the close connection between homology groups of $\text{GL}_n(R)$ for $n \leq 5$ and those of related classifying spaces. The technique to compute the former uses Voronoi’s reduction theory of positive definite and Hermitian quadratic forms, which produces a very large finite cell complex on which $\text{GL}_n(R)$ acts. Our main results are (i) $K_4(\mathbb{Z}[\sqrt{-1}])$ is a finite abelian 3-group, and (ii) $K_4(\mathbb{Z}[(1 + \sqrt{-3})/2])$ is trivial.

1. Introduction

1.1. Statement of results. Let $R$ be the ring of integers of a number field $F$. The goal of this paper is the explicit computation of the torsion in the algebraic $K$-groups $K_n(R)$ for $R$ one of two special imaginary quadratic examples: the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ and the Eisenstein integers $\mathbb{Z}[(1 + \sqrt{-3})/2]$. Our work is in the spirit of Lee–Szczarba [12, 13, 11], Soulé [20, 19], and Elbaz-Vincent–Gangl–Soulé [6, 7], who treated $K_N(\mathbb{Z})$, and Staffeldt [21], who investigated $K_3$ of the Gaussian integers. As in these works, the first step is to compute the cohomology of $\text{GL}_n(R)$ for $n \leq N + 1$; information from this computation is then assembled into information about the $K$-groups following the program in §1.2.

We first state our main intermediate result (Theorem 4.1) that depends only on those computations.

Theorem 1.1. The order of $K_4(\mathbb{Z}[\sqrt{-1}])$ or $K_4(\mathbb{Z}[(1 + \sqrt{-3})/2])$ is not divisible by any prime $> 3$.

We can then improve on this result by invoking work of Rognes–Østvær [18] who in turn heavily use Voevodsky’s celebrated result, the proof of the Milnor Conjecture [22], which implies in the cases at hand that the 2-part in both groups is trivial, and a result of Weibel [25]. Theorem 70, Example 75] which relies on another deep result by Rost and Voevodsky (formerly the Bloch-Kato Conjecture, see [23] and, e.g., a recent Bourbaki talk by J. Riou [16]) to show that $p = 3$ does not divide the order of $K_4(\mathbb{Z}[(1 + \sqrt{-3})/2])$. The actual statement combines Corollary 5.1 and Theorem 5.2 (recall that a 3-group is a group where each element has an order a power of 3).

Theorem 1.2.

1. The group $K_4(\mathbb{Z}[\sqrt{-1}])$ is a finite abelian 3-group.
2. The group $K_4(\mathbb{Z}[(1 + \sqrt{-3})/2])$ is the trivial group.
1.2. **The main steps of the computation.** A classical approach to computing algebraic $K$-groups of number rings $R$ goes back to Quillen [14], and instead shifts the focus to computing the homology (with nontrivial coefficients) of certain arithmetic groups. We briefly outline the method before applying it to our two imaginary quadratic cases.

(i) First, by definition the algebraic $K$-group $K_N(R)$ of a ring $R$ is a particular homotopy group of a topological space associated to $R$: we have $K_N(R) = \pi_{N+1}(BQ(R))$, where $BQ(R)$ is a certain classifying space attached to the infinite general linear group $GL(R)$. In particular $BQ(R)$ is the classifying space of the category $\mathcal{Q}(R)$ of finitely generated $R$-modules. This definition is known as Quillen’s $Q$-construction of algebraic $K$-theory [15].

(ii) Via the Hurewicz homomorphism $\pi_{N+1}(BQ(R)) \to H_{N+1}(BQ(R))$, one replaces the homotopy group by a homology group without losing too much information; more precisely, what may get lost is information about small torsion primes that might be killed by the Hurewicz homomorphism.

(iii) Moreover, using a stability result due to Quillen [15] one can pass from $\mathcal{Q}(R)$ to the category $\mathcal{Q}_{N+1}(R)$ of finitely generated $R$-modules of rank $\leq N + 1$ for sufficiently large $N$. This amounts to passing from $GL(R)$ to the finite-dimensional general linear group $GL_{N+1}(R)$.

(iv) The homology groups to be determined are then $H_n(BQ_n(R))$ for $n \leq N + 1$. Rather than compute these directly, one uses the fact that they can be sandwiched between homology groups of $GL_n(R)$, although the homology must be taken with nontrivial coefficients (more precisely the coefficients are the associated Steinberg module $St_n$).

(v) The standard method to compute the latter homology groups is via Voronoi complexes. These are the chain complexes of certain explicit polyhedral reduction domains of a space of positive definite quadratic (respectively, Hermitian) forms of a given rank for $R = \mathbb{Z}$ (respectively, $R$ imaginary quadratic). The Voronoi complex provides most of the desired information on the homology in question: as in (iv), one might again lose information about small primes—in particular, such information could be hidden in the higher differentials of some equivariant spectral sequence involving the stabilizers of cells in the Voronoi complex. In any case, one can usually find a small bound on the sizes of those primes, which means that one can effectively determine the homology and ultimately the $K$-groups modulo small primes.

(vi) Finally one has, as a starting point, vanishing results for both types of homology groups: for $H_n(BQ_n)$ as in Theorem 3.1 below, and for $H_0(GL_n, St_n)$ as used e.g. by Lee–Szczarba [13].

Altogether, given $N$, then (iv)–(vi) for $n \leq N + 1$, together with (ii), often suffice to give a bound $B$ on the primes involved in the torsion of $K_N(R)$. For some specific number fields—including the ones we will treat here—it then suffices to check “$p$-regularity” for certain primes $p \leq B$ in order to further rule out primes that might contribute to the torsion in $K_N(R)$.

1.3. **Guide to the paper.** For the remainder of the paper, we let $i = \sqrt{-1}$ and $\rho = (1 + \sqrt{-3})/2$, and write $\mathbb{Z}[i]$ and $\mathbb{Z}[\rho]$ for the Gaussian and Eisenstein integers, respectively. In §2, we describe the computation of the Voronoi homology of these two number rings (i.e., step (v) above). Many of these results are taken from
our earlier paper [8], although we go beyond those results here to extract more information for the present application. Then in §3 we work our way backwards, starting from step (vi), and then use the results from steps (vi) and (v) to obtain step (iv), i.e. the groups $H_m(BQ_n(R))$. A key role here is played by Quillen’s stability result (iii) for $BQ_n$, which serves as a stopping criterion. Next in §4 we treat step (ii), i.e. the potential primes entering the kernel of the Hurewicz homomorphism. This will then imply our intermediate Theorem above. Finally in §5 we consider $p$-regularity for $p = 2$ for both rings $R$ and for $p = 3$ in the case of $K_4(\mathbb{Z}[p])$ to end up with the information we need for step (i), and which allows us to deduce our main results (Corollary 5.1 and Theorem 5.2).

We conclude with a few remarks about earlier results on the $K$-groups in Theorem 1.2. An at the time conditional determination of the groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[p])$ (and two others) had been given by Kolster [10], who combined a “relative higher class number formula” with Rognes’s result [17] that $K_4(\mathbb{Z})$ is trivial, together with the Quillen-Lichtenbaum conjecture for all odd primes; the latter is now also a consequence of the result by Rost and Voevodsky alluded to above.

2. Homology of Voronoi complexes

We first collect the results from [8] concerning the Voronoi complexes attached to $\Gamma = \text{GL}_n(\mathbb{Z}[i])$ or $\Gamma = \text{GL}_n(\mathbb{Z}[p])$; this is the necessary information needed for step (v) from §1.2 above. For more details about these computations, including background about how the computations are performed, we refer to [8].

Let $F$ be an imaginary quadratic field with ring of integers $R$. The group $\text{GL}_n(R)$ acts on the symmetric space $X_n = \text{GL}_n(\mathbb{C})/U(n)$ and on a certain Satake compactification $X^*_n$. The space $X_n$ can be realized as the quotient of the cone of rank $n$ positive definite Hermitian matrices $C_n$ modulo homotheties. The (partial) compactification $X^*_n$ can also be constructed from $C_n$. One first enlarges the open cone $C_n$ to a bigger domain $C'_n$ by adjoining boundary components corresponding to the positive semi-definite Hermitian forms with an $F$-rational nullspace. Then $X^*$ is the quotient of $C'_n$ by homotheties. The group $\Gamma$ also acts on $X^*_n$, and the quotient $\Gamma\backslash X^*_n$ is compact and Hausdorff. Let $\partial X^*_n$ be the boundary $X^*_n \backslash X_n$.

A generalization—due to Ash [2] Chapter II and Koecher [9]—of the polyhedral reduction theory of Voronoi [24] yields a $\Gamma$-equivariant explicit decomposition of $X^*_n$ into cells. Moreover, there are only finitely many cells modulo $\Gamma$. Let $\Sigma_p = \Sigma_p(\Gamma)$ be a set of representatives of the $\Gamma$-inequivalent cells in dimension $p$ that meet the interior $X_n$, and let $\Sigma_p = \Sigma_p(\Gamma)$ be the subset of representatives of the $\Gamma$-inequivalent orientable cells in this dimension; here we call a cell orientable if all the elements in its stabilizer group preserve its orientation. (Note that in our discussion of these decompositions and associated algebraic constructions, we often use $p$ as an index, not as a prime.) One can form a chain complex $\text{Vor}_n$, the Voronoi complex, and one can prove that modulo small primes the homology of this complex is the homology $H_i(\Gamma, St_n)$, where $St_n$ is the rank $n$ Steinberg module (cf. [4]). In particular, let $S_k$ denote the Serre class of finite abelian groups, the order of which has only prime factors less than or equal to $k$.

**Theorem 2.1** ([8] Theorem 3.7]). Let $b$ be an upper bound on the torsion primes for $\text{GL}_n(R)$. Then $H_m(\text{Vor}_n) \cong H_{m-n+1}(\text{GL}_n(R), St_n)$ modulo $S_b$. 

2.1. Voronoi data for \( R = \mathbb{Z}[i] \). We now give results for the Voronoi complexes and their homology in the cases relevant to our paper. This subsection treats the Gaussian integers; in §2.2 we treat the Eisenstein integers.

**Theorem 2.2 ([21]).**

1. There is one Voronoi cell for \( \text{GL}_2(\mathbb{Z}[i]) \) in each of the dimensions 1, 2, and 3. Only the cell in dimension 3 is orientable.

2. The number of Voronoi cells for \( \text{GL}_3(\mathbb{Z}[i]) \) is as follows:

<table>
<thead>
<tr>
<th>( p )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_p(\text{GL}_2(\mathbb{Z}[i])) )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Sigma_p(\text{GL}_3(\mathbb{Z}[i])) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Theorem 2.3 ([8, Table 12]).** The number of Voronoi cells for \( \text{GL}_4(\mathbb{Z}[i]) \) is as follows:

<table>
<thead>
<tr>
<th>( p )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_p(\text{GL}_4(\mathbb{Z}[i])) )</td>
<td>4</td>
<td>10</td>
<td>33</td>
<td>98</td>
<td>258</td>
<td>501</td>
<td>704</td>
<td>628</td>
<td>369</td>
<td>130</td>
<td>31</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>( \Sigma_p(\text{GL}_4(\mathbb{Z}[i])) )</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>48</td>
<td>189</td>
<td>435</td>
<td>639</td>
<td>597</td>
<td>346</td>
<td>120</td>
<td>22</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

We remark that for \( n = 3 \) the Voronoi complexes and their homology ranks were originally computed by Staffordt [21], who even distilled the 3-part for each homology group. After calculating the differentials one obtains, in agreement with Staffordt’s results, the following homology groups:

**Theorem 2.4 ([21, Theorems IV, 1.3 and 1.4, p.785]).**

1. In rank 2, modulo \( S_3 \) we have

\[
H_m(\text{GL}_2(\mathbb{Z}[i]), S_2) = \begin{cases} 
\mathbb{Z} & \text{if } m = 2, \\
0 & \text{otherwise}.
\end{cases}
\]

2. In rank 3, modulo \( S_3 \) we have

\[
H_m(\text{GL}_3(\mathbb{Z}[i]), S_3) = \begin{cases} 
\mathbb{Z} & \text{if } m = 2, 3, 6, \\
0 & \text{otherwise}.
\end{cases}
\]

In particular, from the above theorem we deduce that the only possible torsion primes for \( H_m(\text{GL}_n(\mathbb{Z}[i]), S_n) \) for \( n = 2, 3 \) are the primes 2 and 3.

For \( n = 4 \), Table 12 of [8] shows that, for \( \text{GL}_4(\mathbb{Z}[i]) \), the elementary divisors of all the differentials in the Voronoi complex are supported on primes \( \leq 5 \). In fact, closer examination of this table reveals the following:

**Theorem 2.5 ([8, Theorem 7.2 and Table 12]).** Modulo \( S_5 \) we have

\[
H_m(\text{GL}_4(\mathbb{Z}[i]), S_4) = \begin{cases} 
\mathbb{Z}^2 & \text{if } m = 5, \\
\mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, the only possible degrees where 5-torsion can occur are \( m = 1, 6 \) or \( m > 9 \).
We need to discuss the potential 5-torsion for \( H_m(\text{GL}_4(\mathbb{Z}[i]), \text{St}_4) \). While there is 5-torsion in \( H_{10} \), and possibly further 5-torsion in \( H_m \) for \( m \geq 6 \), we will show that for degree \( m = 1 \) (the only relevant degree for the \( K \)-groups we consider) the group \( H_1 \) contains no 5-torsion (Proposition 2.6).

In order to analyze \( H_1(\text{GL}_4(\mathbb{Z}[i]), \text{St}_4) \) more closely, we will need the following notions and results. According to [5, VII.7] there is a spectral sequence converging to the equivariant homology groups \( H^1 \), that for degree \( m \) is 5-torsion in \( H^m(\mathbb{Z}[i], \partial X^{\ast}_n; \mathbb{Z}) \) of the homology pair \( (X^{\ast}_n, \partial X^{\ast}_n) \), and such that

\[
E^1_{pq} = \bigoplus_{\sigma \in \Sigma^1_p} H_q(\Gamma_{\sigma}, \mathbb{Z}_{\sigma}),
\]

where \( \mathbb{Z}_{\sigma} \) is the orientation module of the cell \( \sigma \).

**Proposition 2.6.** The group \( H_1(\text{GL}_4(\mathbb{Z}[i]), \text{St}_4) \) is zero modulo \( \mathcal{S}_3 \).

**Proof.** Since \( H_1(\text{GL}_4(\mathbb{Z}[i]), \text{St}_4) \) is a subquotient of \( \bigoplus_{p+q=4} E^1_{p,q} \), we consider the individual summands \( E^1_{p,4-p} \) for \( p = 0, \ldots, 4 \):

- Since there are no cells in \( \Sigma^1_p \) for \( p \leq 2 \), we have \( E^1_{0,4} = E^1_{1,3} = E^1_{2,2} = 0 \).
- Consider now \( p = 3 \). There are four cells in \( \Sigma^1_3 \), and for each of them the index 2 subgroup acting trivially on the orientation module has an abelianization \( \mathbb{Z}/\mathbb{Z} \) up to 2-groups. Thus in particular we have

\[
E^1_{3,1} = \bigoplus_{\sigma \in \Sigma^1_3} H_1(\text{Stab}(\sigma, \mathbb{Z}_{\sigma})) \in \mathcal{S}_3,
\]

and this term contains no 5-torsion.
- Finally, for \( p = 4 \), there is only one cell (out of ten) in \( \Sigma^1_4 \), denoted by \( \sigma^4_1 \), that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in \( H_1(\text{Stab}(\sigma^4_1), \mathbb{Z}) \) (here \( \mathbb{Z} \) is the orientation module \( \mathbb{Z}_{\sigma^4_1} \)). Indeed, the order-preserving subgroup \( K_1 \) of \( \text{Stab}(\sigma^4_1) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times A_5 \), where \( A_5 \) is the alternating group of five letters, with abelianization \( H_1(\text{Stab}(\sigma^4_1), \mathbb{Z}) = H_1(K_1, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \), which lies in \( \mathcal{S}_3 \). Thus there can be no 5-torsion from here, and this completes the proof of the proposition.

2.2. **Voronoi homology data for** \( R = \mathbb{Z}[\rho] \). Now we turn to the Eisenstein case.

**Theorem 2.7** ([8] Tables 1 and 11).

1. There is one Voronoi cell for \( \text{GL}_2(\mathbb{Z}[\rho]) \) in each of the dimensions 1, 2, and 3. Only the cell in dimension 3 is orientable.

2. The number of Voronoi cells for \( \text{GL}_3(\mathbb{Z}[\rho]) \) is as follows:

\[
\begin{array}{cccccccc}
\ p \ & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
|\Sigma_p(\text{GL}_3(\mathbb{Z}[\rho]))^\ast| \ & 1 & 2 & 3 & 4 & 3 & 2 & 2 \\
|\Sigma_p(\text{GL}_3(\mathbb{Z}[\rho]))| \ & 0 & 0 & 1 & 2 & 1 & 1 & 2 \\
\end{array}
\]

3. The number of Voronoi cells for \( \text{GL}_4(\mathbb{Z}[\rho]) \) is as follows:

\[
\begin{array}{cccccccccccccccc}
\ p \ & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
|\Sigma_p(\text{GL}_4(\mathbb{Z}[\rho]))^\ast| \ & 2 & 5 & 12 & 34 & 82 & 166 & 277 & 324 & 259 & 142 & 48 & 15 & 5 \\
|\Sigma_p(\text{GL}_4(\mathbb{Z}[\rho]))| \ & 0 & 0 & 0 & 8 & 50 & 129 & 228 & 286 & 237 & 122 & 36 & 10 & 5 \\
\end{array}
\]
Theorem 2.8 ([8] Theorems 7.1 and 7.2]).

1. In rank 2, modulo $S_3$ we have

$$H_m(\text{GL}_2(\mathbb{Z}[\rho]), St_2) = \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise}. \end{cases}$$

2. In rank 3, modulo $S_3$ we have

$$H_m(\text{GL}_3(\mathbb{Z}[\rho]), St_3) = \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise}. \end{cases}$$

3. In rank 4, modulo $S_3$ we have

$$H_m(\text{GL}_4(\mathbb{Z}[\rho]), St_4) = \begin{cases} \mathbb{T}^2 & \text{if } m = 5, \\ \mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Since the ranks of the homology groups in question have been computed in [8], we only have to consider the torsion in the respective groups. For fixed $n$, any torsion prime of the homology groups $H_m(\text{GL}_n(\mathbb{Z}[\rho]), St_n)$ must either divide the order of the stabilizer of some cell in $\Sigma_p$ for appropriate $p$, or must divide an elementary divisor of the differentials in the corresponding Voronoi complex. We consider these two possibilities in turn.

First we consider the stabilizers. For ranks $n = 2, 3$, all stabilizers of cells in $\Sigma_p^*$ lie in $S_3$. For rank $n = 4$, the prime 5 is the only torsion prime $> 3$ occurring for stabilizer orders in $\Sigma_p^*$, more precisely it occurs for $p = 9$ (two cells), $p = 14$ (two cells) and $p = 15$ (one cell).

Next we consider elementary divisors. In rank $n = 2$, the elementary divisors occurring are all even, and apart from $m = 2$, where $H_2(\text{GL}_2(\mathbb{Z}[\rho]), St_2) = H_3(\text{Vor}_*) = \mathbb{Z}$ modulo $S_3$, we have $H_m(\text{GL}_2(\mathbb{Z}[\rho]), St_2) = H_{m+1}(\text{Vor}_*) = 0$ modulo $S_3$. In rank $n = 3$, the only non-trivial elementary divisor for any of the differentials involved is 9, arising from $d_8^1 : E_{8,0}^1 \rightarrow E_{7,0}^1$. Moreover, we get $H_m(\text{GL}_3(\mathbb{Z}[\rho]), St_3) = H_{m+2}(\text{Vor}_*) = \mathbb{Z}$ modulo $S_3$ for $m = 2, 3$ or 6, and is zero otherwise. Finally, for rank $n = 4$, the only torsion prime $> 3$ for the homology groups $H_{m+3}(\text{Vor}_*)$ is $p = 5$, which divides the elementary divisor 15 of $d_{14}^1$. This completes the proof.

As before, a refined analysis of the case $n = 4$ shows that $H_1$ contains no 5-torsion:

Proposition 2.9. Modulo $S_3$, we have

$$H_1(\text{GL}_4(\mathbb{Z}[\rho]), St_4) = 0.$$

Proof. The argument is very similar to that of the proof of Proposition 2.6. In rank 4, we have that $H_1(\text{GL}_4(\mathbb{Z}[\rho]), St_4)$ is a subquotient of

$$\bigoplus_{p+q=4} E_{p,q}^1 = \bigoplus_{p+q=4} \bigoplus_{\sigma \in \Sigma_p^*} H_{\sigma, 1}(\text{Vor}_*, \mathbb{Z}).$$

We consider each of these summands in turn.
If \( p \leq 2 \), then there are no cells of dimension \( p \) to worry about. For \( p = 3 \), there are two cells in \( \Sigma_3 \), with stabilizer in \( S_3 \), and hence
\[
E_{3,1}^1 = \bigoplus_{\sigma \in \Sigma_3} H_1(\text{Stab}(\sigma), \mathbb{Z}) \in S_3.
\]
Finally suppose \( p = 4 \). Then \( E_{4,0}^1 = 0 \mod S_2 \), as none of the 5 classes in dim 4 is orientable. Thus modulo \( S_3 \) all summands in (7) vanish, which completes the proof.

\[ \square \]

3. Vanishing and Sandwiching

In this section, we carry out the sandwiching argument (step (iv) of §1.2). As a first step we invoke a vanishing result for homology groups for \( BQ_1 \) due to Quillen [14, p.212]. In our cases this result boils down to the following statement:

**Theorem 3.1.** For the rings \( R = \mathbb{Z}[i] \) and \( \mathbb{Z}[\rho] \), we have
\[
H_n(BQ_1) = 0 \quad \text{whenever } n \geq 3.
\]

For \( R = \mathbb{Z}[i] \) a slightly stronger result is proved in [21, Lemma I.1.2]. However, we will not need this stronger result for \( \mathbb{Z}[i] \), or its analogue for \( \mathbb{Z}[\rho] \).

Using our homology data from §2 and Theorem 3.1, we can get for both rings \( R = \mathbb{Z}[i] \) and \( R = \mathbb{Z}[\rho] \) the following result:

**Proposition 3.2.** \( H_5(BQ) = \mathbb{Z} \) modulo \( S_3 \).

**Proof.** We will successively determine \( H_5(BQ_j) \) for \( j = 1, \ldots, 5 \) and then identify the last group via stability with \( H_5(BQ) \). For this, we will combine results from §2 with Quillen’s long exact sequence for different \( r \), given by
\[
\cdots \longrightarrow H_n(BQ_{r-1}) \longrightarrow H_n(BQ_r) \longrightarrow H_{n-r}(\text{GL}_r, S_t_r) \longrightarrow H_{n-1}(BQ_{r-1}) \longrightarrow \cdots.
\]

The case \( j = 1 \). By Theorem 3.1 we have \( H_n(BQ_1) = 0 \) for \( n \geq 3 \).

The case \( j = 2 \). From the above sequence (8) for \( r = 2 \), we get
\[
\begin{align*}
H_5(BQ_1) &\longrightarrow H_5(BQ_2) \longrightarrow H_5(\text{GL}_2, S_t_2) \longrightarrow H_4(BQ_1), \\
&= 0
\end{align*}
\]
whence \( H_5(BQ_2) = 0 \mod S_3 \) by (1) and (3).

The case \( j = 3 \). Now we invoke another result of Staffeldt’s who showed (see [21, proof of Theorem I.1.1]) that
\[
H_4(BQ_2) = H_4(BQ_3) = \mathbb{Z} \mod S_3.
\]

From (8) for \( r = 3 \) we get the exact sequence, working mod \( S_3 \),
\[
H_5(BQ_2) \longrightarrow H_5(BQ_3) \longrightarrow H_5(\text{GL}_3, S_t_3) \longrightarrow H_4(BQ_3) \longrightarrow H_4(BQ_3) \longrightarrow H_4(\text{GL}_3, S_t_3).
\]

Since the leftmost group \( H_5(BQ_3) \) vanishes modulo \( S_3 \) by the case \( j = 2 \), this sequence implies that \( H_5(BQ_3) = \mathbb{Z} \mod S_3 \).

The case \( j = 4 \). Moreover, since \( H_5(\text{GL}_4, S_t_4) = H_4(\text{GL}_4, S_t_4) = 0 \mod S_3 \) by Theorem 2.5 and Proposition 2.6, the sequence (8) for \( r = 4 \) gives in a similar way that
\[
H_5(BQ_4) = H_5(BQ_3) = \mathbb{Z} \mod S_3.
\]
The case \( j = 5 \). This is the most complicated of all the cases to handle. Note that \( B\mathbb{Q} \) is an \( H \)-space which implies that \( H_\ast(B\mathbb{Q}) \otimes \mathbb{Q} \) is the enveloping algebra of \( \pi_\ast(B\mathbb{Q}) \otimes \mathbb{Q} \). We know that \( K_0(\mathbb{Z}[i]) = \mathbb{Z} \), \( K_1(\mathbb{Z}[i]) = \mathbb{Z}/2 \) and \( K_2(\mathbb{Z}[i]) = 0 \) \([3\] Appendix\) as well as \( K_3(\mathbb{Z}[i]) = \mathbb{Z} \oplus \mathbb{Z}/24 \) (cf. Weibel [25], Theorem 73 in combination with Example 28), so modulo \( S_3 \) we have that

\[
\pi_1(B\mathbb{Q}) \otimes \mathbb{Q} = K_0(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q},
\]
that \( \pi_2(B\mathbb{Q}) \otimes \mathbb{Q} = \pi_3(B\mathbb{Q}) \otimes \mathbb{Q} = 0 \), and that

\[
\pi_4(B\mathbb{Q}) \otimes \mathbb{Q} = K_3(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q}.
\]

Hence \( H_3(B\mathbb{Q}) \otimes \mathbb{Q} \) contains the product of \( \pi_1(B\mathbb{Q}) \otimes \mathbb{Q} \) by \( \pi_4(B\mathbb{Q}) \otimes \mathbb{Q} \) and so its dimension is at least 1.

The stability result foretold in step (iii) of §1.2 (resulting for a Euclidean domain \( \Lambda \) from \( H_0(\text{GL}_n(\Lambda), S_4) = 0 \) for \( n \geq 3 \), [12] Corollary to Theorem 4.1], now implies that one has \( H_3(B\mathbb{Q}) = H_3(B\mathbb{Q}_5) \). By the above we get that the rank of \( H_3(B\mathbb{Q}_5) = H_3(B\mathbb{Q}) \) is at least 1.

Therefore, invoking yet again Quillen’s exact sequence (8), this time for \( r = 5 \), and using the above result that \( H_3(B\mathbb{Q}_4) \) is equal to \( \mathbb{Z} \) modulo \( S_3 \), we deduce from

\[
\begin{align*}
H_5(B\mathbb{Q}_4) & \longrightarrow H_5(B\mathbb{Q}_5) \longrightarrow H_0(\text{GL}_5, S_5) \\
\text{mod } \mathbb{Z}, \text{ by (10)} & \Rightarrow 0
\end{align*}
\]
that \( H_5(B\mathbb{Q}) = H_5(B\mathbb{Q}_5) \) must be equal to \( \mathbb{Z} \) modulo \( S_3 \) as well. Thus \( H_5(B\mathbb{Q}) \) cannot contain any \( p \)-torsion with \( p > 3 \). \( \square \)

4. Relating \( K_4(\mathbb{O}) \) and \( H_5(B\mathbb{Q}(\mathbb{O})) \) via the Hurewicz homomorphism

It is well known that for a number ring \( R \) the space \( B\mathbb{Q}(R) \) is an infinite loop space, hence a theorem due to Arlettaz [11] Theorem 1.5] shows that the kernel of the corresponding Hurewicz homomorphism \( K_4(R) = \pi_4(B\mathbb{Q}) \rightarrow H_5(B\mathbb{Q}) \) is certainly annihilated by 144 (cf. Definition 1.3 in loc.cit., where this number is denoted \( R_5 \)). Thus \( K_4(R) \) lies in \( S_3 \).

Therefore this Hurewicz homomorphism is injective modulo \( S_3 \). For \( R = \mathbb{Z}[i] \) or \( \mathbb{Z}[\rho] \), Proposition 3.2 implies that \( H_5(B\mathbb{Q}) \) contains no \( p \)-torsion for \( p > 3 \). After invoking Quillen’s result that \( K_{2n}(R) \) is finitely generated and Borel’s result that the rank of \( K_{2n}(R) \) is zero for any number ring \( R \) and \( n > 0 \), we obtain the following intermediate result:

**Theorem 4.1.** The groups \( K_4(\mathbb{Z}[i]) \) and \( K_4(\mathbb{Z}[\rho]) \) lie in \( S_3 \).

5. \( p \)-regularity of \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\rho] \)

For our final conclusion, we use \( p \)-regularity for \( p = 2, 3 \) for the rings \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\rho] \) to rule out more torsion in \( K_4 \). Recall that a ring of integers \( R \) is called \( p \)-regular for some prime \( p \) if \( p \) is not split in \( R \) and if the narrow class number of \( R[\sqrt{1/p}] \) is coprime to \( p \).

First we consider 2-regularity. Rognes and Østvær [18] show that the group \( K_{2n}(R) \) has trivial 2-part if \( R \) is the ring of integers of a 2-regular number field \( F \). This applies to both imaginary quadratic fields we consider, since 2 ramifies (respectively is inert) in \( \mathbb{Q}(i) \) (respectively \( \mathbb{Q}(\rho) \)), and both fields have class number 1 and no real places. In particular, \( |K_4(\mathbb{Z}[i])| \) and \( |K_4(\mathbb{Z}[\rho])| \) must both be odd. Combining this with Theorem 4.1, we obtain the following result:
Corollary 5.1. The groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ are 3-groups.

Next we consider 3-regularity. For $R = \mathbb{Z}[i]$, we are unfortunately unable to go further, but following a suggestion of Soulé we can say more for $R = \mathbb{Z}[\rho]$; we can apply the fact that if $\ell$ is a regular odd prime, then there is no $\ell$-torsion in $K_{2\ell}(\mathbb{Z}[\zeta_{\ell}])$, where $\zeta_{\ell}$ is a primitive $\ell$th root of unity (cf. [25, Example 75]). Since 3 is a regular prime (the first irregular prime is 37), and since $\mathbb{Z}[\rho] = \mathbb{Z}[\zeta_3]$, we obtain the following result:

Theorem 5.2. The group $K_4(\mathbb{Z}[\rho])$ is trivial.

References


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