HENRY JACOB MATHEMATICS COMPETITION

SAMPLE PROBLEMS FROM PREVIOUS COMPETITIONS

No calculators are allowed!

(1) (a) If \( f(x) = x^2 e^x \), find a formula for the \( n \)th derivative of \( f \).
(b) Prove that your formula is correct.

(2) What is the smallest possible area of a triangle whose sides are formed by the positive \( x \) and \( y \) axes and a line through the point \((1, 2)\)?

(3) Prove that for any positive integer \( n \), the expression \( 1110^n + 1102^n - 200^n - 10^n \) is divisible by 2002.

(4) The \((5, 12, 13)\) right triangle has area and perimeter both equal to 30. Find all right triangles \( T \) with integral sides such that the area of \( T \) equals the perimeter of \( T \).

(5) Find
\[
\int_{-1}^{1} \frac{e^{1/x}}{x^2(1 + e^{1/x})} \, dx.
\]

(6) Let \( R(t) \) be the area of the region bounded by the \( y \)-axis, a positive continuous function \( f(x) \), a negative continuous function \( g(x) \), and the line \( x = t^4 \). Compute \( R'(2) \).

(7) A function is odd if \( f(-x) = -f(x) \). Prove that \( f(x) = \ln(x + \sqrt{x^2 + 1}) \) is odd.

(8) Show that there are infinitely many squares that are the sum of a square and a prime.

(9) Let \( g(x) \) be the greatest integer less than or equal to \( x \). For example \( g(\pi) = 3 \) and \( g(-\pi) = -4 \). Sketch the graphs of the following:
(a) \( y = g(x) \)
(b) \( g(y) = g(x) \)
(c) \( g(y) = |g(x)| \)

(10) Find a polynomial \( P \) with \( P(1) = 1, P(2) = 2, P(3) = 2003 \).

(11) An “hourglass” is formed by rotating the graph of \( y = e^x \) about the line \( y = x \). Find the largest tangent sphere that can be passed through the neck of the hourglass.

(12) (a) Show that \( x/y + y/z + z/x = 1 \) has no solution in positive integers.
(b) Find a solution to \( x/y + y/z + z/x = 5 \) in positive integers.