Math 455: Discrete Structures
Prof. Paul Gunnells
TA: Jennifer Li

Homework 7 Solutions

§2.6

2) By the example on p.165, the generating function for choosing six beads without any conditions is:

$$G(x) = \underbrace{(x^0+x^1+x^2+x^3)}_{\text{3 red beads}}\underbrace{(x^0+x^1+x^2+x^3+x^4)}_{\text{4 blue beads}}\underbrace{(x^0+x^1+x^2+x^3+x^4+x^5)}_{\text{5 green beads}}$$

because we have 3 red beads, 4 blue beads, and 5 green beads total.

But with the three conditions, we can rewrite the generating function as:

$$G(x) = (\cancel{x^0}+x^1+x^2+x^3)\underbrace{(\cancel{x^0}+x^1+\cancel{x^2}+x^3+\cancel{x^4})}_{\text{Because we want an odd \# of blue beads}}\underbrace{(x^0+\cancel{x^1}+x^2+\cancel{x^3}+x^4+\cancel{x^5})}_{\text{Because we want an even \# of green beads.}}$$

Because we want at least one red bead

$= (x+x^2+x^3)(x+x^3)(1+x^2+x^4)$

$= x^2+x^3+3x^4+x^5+4x^6+x^7+3x^8+x^{10}$

∴ there are 4 ways to choose six beads with the three given conditions. □

3 (a) $$G(x) = \underbrace{(1+x+x^2+\cdots+x^{10})}_{\text{10 red}}\underbrace{(1+x+\cdots+x^8)}_{\text{8 blue}}\underbrace{(1+x+\cdots+x^{11})}_{\text{11 green}}$$

$x^0 = 1$

(b) For an even # of red, an odd # of blue, and a prime # of green:

$$G(x) = (1+x^2+x^4+x^6+x^8+x^{10})(x+x^3+x^5+x^7)(x^2+x^3+x^5+x^7+x^{11})$$

(c) For two red, at least five blue, and at most four green:

$$G(x) = x^2(x^5+x^6+x^7+x^8)(1+x+x^3+x^4)$$

□

§2.6.1

2) For a triple deck we have the following possibilities:

All 5 cards are different: $\binom{52}{5}$

One card appears twice: $\binom{52}{4}\cdot\binom{4}{1}$

One card appears three times: $\binom{52}{3}\cdot\binom{3}{1}$

Two cards appear twice: $\binom{52}{3}\cdot\binom{3}{2}$

One card appears twice, one card appears three times: $\binom{52}{2}\cdot\binom{2}{1}$

The sum of these give the total number of different five-card hands: 3817112.

For a quadruple deck, we have the same possibilities as above, plus one more:

One card appears four times: $\binom{52}{2}\cdot\binom{2}{1}$

So the number of different five-card hands from a quadruple deck is 3819764.

$\square$

3) For one deck: We must have all six cards different; this is a total of

$\binom{52}{6}$ possibilities

For two decks: either all cards are different, or two cards are the same, giving a total of $\binom{52}{6}+\binom{52}{5}\cdot\binom{5}{1}$ possibilities

For three decks: Same possibilities as two decks, plus the case where there are three of the same cards:

$$\binom{52}{4}\cdot\binom{4}{1}$$

and only two cards appear twice:

$$\binom{52}{4}\cdot\binom{4}{2}$$

and three cards appear twice:

$$\binom{52}{3}$$

§ 2.6.1

3) (cont.ed)

and one card appears twice, and another appears three times:

$$\binom{52}{3}\binom{3}{2}\cdot\binom{2}{1}$$

This gives a total of

$$\binom{52}{6}+\binom{52}{5}\binom{5}{1}+\binom{52}{4}\binom{4}{1}+\binom{52}{4}\binom{4}{2}+\binom{52}{3}+\binom{52}{3}\binom{3}{2}\binom{2}{1}$$ possibilities

For four decks: Same as three decks, plus the case where

One card appears four times and the other two are distinct:

$$\binom{52}{3}\cdot\binom{3}{1}$$

and the case where one card appears four times and one card appears twice:

$$\binom{52}{2}\binom{2}{1}$$

giving a total of

$$\binom{52}{6}+\binom{52}{5}\binom{5}{1}+\binom{52}{4}\binom{4}{1}+\binom{52}{4}\binom{4}{2}+\binom{52}{3}+\binom{52}{3}\binom{3}{2}\binom{2}{1}+\binom{52}{3}\binom{3}{1}+\binom{52}{2}\binom{2}{1}$$

possibilities.

For five decks: Same as four decks plus one more case, which is one card appears five times

$$\binom{52}{2}\cdot\binom{2}{1}$$

For six decks: Same as five decks plus the case where all six cards are the same

$$\binom{52}{1}$$

□

§2.6.3

2) The generating function is

$$G(x) = \underbrace{(1+x+x^2+x^3+x^4+x^5+\cdots)}_{\text{pennies}}\underbrace{(1+x+x^2+x^3+x^4+x^5+\cdots)}_{\text{nickels}}\underbrace{(\text{ same })}_{\text{dimes}}\cdot\underbrace{(\text{" \quad "})}_{\text{quarters}}\underbrace{(\text{" \quad "})}_{\text{1/2 dollars}}\underbrace{(\text{" \quad "})}_{\text{dollar coins}}$$

The number of ways to select 100 coins is the coefficient of x^{100} in $G(x)$

$\square$

3) By the formula given on p. 172,

$$\begin{aligned} d_{10m} &= n_{10m} + d_{10(m-1)} \\ &= n_{10m} + (n_{10(m-1)} + d_{10(m-1)-10}) \\ &= n_{10m} + n_{10(m-1)} + d_{10(m-2)} \\ &= \cdots \\ &= n_{10m} + n_{10(m-1)} + \cdots + n_0 \end{aligned}$$

Now $p_k = 1$ for all k, so by the second to last formula on p. 172:

$$n_k = \begin{cases} p_k & \text{if } 0 \le k \le 4, \\ p_k + n_{k-5} & \text{if } k \ge 5. \end{cases}$$

we have

$$\left.\begin{aligned} n_1 &= p_1 = 1 \\ n_2 &= p_2 = 1 \\ n_3 &= p_3 = 1 \\ n_4 &= p_4 = 1 \end{aligned}\right\} \quad 0 \le k \le 4$$

$$\left.\begin{aligned} n_5 &= p_5 + n_0 = 1 + 0 = 1 \\ n_6 &= p_6 + n_1 = 1 + 1 = 2 \\ n_7 &= p_7 + n_2 = 1 + 1 = 2 \\ n_8 &= p_8 + n_3 = 1 + 1 = 2 \\ &\vdots \\ n_{11} &= p_{11} + p_6 = 1 + 2 = 3 \end{aligned}\right\} \quad k \ge 5$$

$\vdots \qquad \vdots$

So $\{n_k\} = \{1,1,1,1,\ 2,2,2,2,2,\ 3,3,3,3,3,\ldots\}$

Then $n_k = \left\lfloor \frac{k}{5} \right\rfloor + 1$, so $n_{10i} = \left\lfloor \frac{10i}{5} \right\rfloor + 1 = \lfloor 2i \rfloor + 1 = 2i + 1$

§2.6.3

3 (cont.ed)

Then, $d_{10m} = n_{10m} + n_{10(m-1)} + \cdots + n_1 + n_0$

$$= (2m+1) + (2(m-1)+1) + \cdots + (2+1) + (0+1)$$

$$= 2\left(\sum_{i=1}^{m} i\right) + m + 1$$

$$= 2 \cdot \frac{m(m+1)}{2} + m + 1$$

$$= (m+1)^2$$

$\square$

§2.6.4

2) $G(x) = \sum_{k \geq 0} a_k x^k$

$$= a_0 + a_1 x + \sum_{k \geq 2} a_k x^k$$

$$= 0 + 8x + \sum_{k \geq 2} (2a_{k-1} + 3a_{k-2}) x^k$$

$$= 8x + \sum_{k \geq 2} 2a_{k-1} x^k + \sum_{k \geq 2} 3a_{k-2} x^k$$

$$= 8x + 2x \sum_{k \geq 2} a_{k-1} x^{k-1} + 3x^2 \sum_{k \geq 2} a_{k-2} x^{k-2}$$

$$= 8x + 2x \sum_{k \geq 1} a_k x^k + 3x^2 \sum_{k \geq 0} a_k x^k$$

$\Rightarrow G(x) = 8x + 2x\,G(x) + 3x^2 G(x)$

$\Rightarrow G(x)(1 - 2x - 3x^2) = 8x$

$\Rightarrow G(x) = \dfrac{8x}{(x+3)(1-3x)} = \dfrac{A}{x+3} + \dfrac{B}{1-3x}$

$\Rightarrow 8x = A(1-3x) + B(x+3)$

$\Rightarrow A = -12/5$ and $B = 4/5$

$\Rightarrow G(x) = -\dfrac{12}{5(x+3)} + \dfrac{4}{5(1-3x)}$

$\square$

§2.6.4

3) $G(x) = \sum_{k\geq 0} a_k x^k$

$$= a_0 + a_1 x + \sum_{k\geq 2} a_k x^k$$

$$= 0 + 5x + \sum_{k\geq 2} (a_{k-1} + 6a_{k-2}) x^k$$

$$= 5x + \sum_{k\geq 2} a_{k-1} x^k + 6 \sum_{k\geq 2} a_{k-2} x^k$$

$$= 5x + x \sum_{k\geq 1} a_k x^k + 6x^2 \sum_{k\geq 0} a_k x^k$$

$$\Rightarrow G(x) = 5x + xG(x) + 6x^2 G(x)$$

$$\Rightarrow G(x)(1 - x - 6x^2) = 5x$$

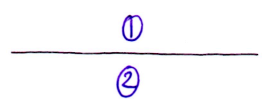
$$\Rightarrow G(x) = \frac{5x}{1 - x - 6x^2}$$

$$\Rightarrow G(x) = \frac{1}{1-3x} - \frac{1}{2x+1}$$

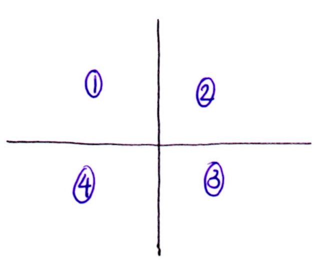
$\square$

§2.6.5

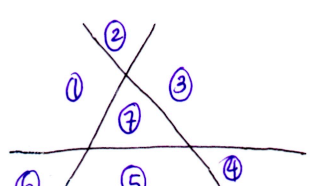
1) The maximal number of regions is created when no lines are parallel.
Let l_k be the maximal number of regions in the plane separated by k straight lines.
Then $l_1 = 2$:



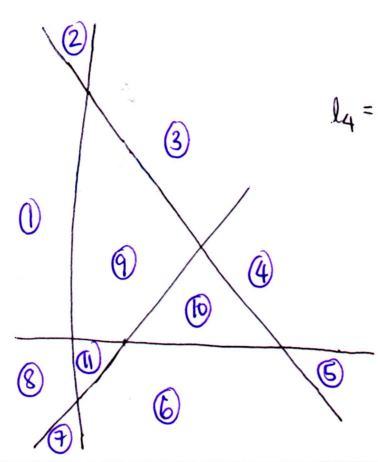
$l_2 = 4$:



$l_3 = 7$:



To get l_{k+1}, add a line that intersects all lines :



$l_4 = 11$

§2.6.5

1) (cont.ed)

Then
$$
\begin{aligned}
l_{k+1} &= l_k + (k+1) \\
&= (l_{k-1} + (k-1)+1) + k+1 \\
&= l_{k-1} + k + (k+1) \\
&= \cdots \\
&= 1 + (1+2+\cdots+k+(k+1)) \\
&= 1 + \frac{k(k+1)}{2}
\end{aligned}
$$

$\square$

7) Recurrence relation:

Suppose we have a sequence of length $n-1$, consisting of only 0's and 1's, such that there are no three adjacent 1's.

Suppose we want to add a 0 or a 1 to create a sequence of length n.

There are 2 possibilities:

1) We add a 0 to the end

$$\underbrace{\_\ \_\ \cdots\ \_}_{n-1 \text{ seq.}}\ \underset{\substack{\uparrow \\ \text{add}}}{0}$$

This is always OK, since it will never introduce 1 1 1 into the seq.

2) We add a 1 at the end

$$\underbrace{\_\ \_\ \cdots\ \_}_{n-1 \text{ seq.}}\ \underset{\substack{\uparrow \\ \text{add}}}{1}$$

There are 2 cases:

2.1) The $(n-1)$ seq. ends in a 0:

$$\underbrace{\_\ \_\ \cdots\ \underline{0}}_{n-1}\ \underset{\substack{\uparrow \\ \text{added}}}{1} \quad = \quad \underbrace{\_\ \_\ \cdots\ \_}_{n-2}\ \underline{0}\ \underline{1}$$

In this case everything is OK

2.2) The $(n-1)$ seq. ends in a 1:

$$\underbrace{\_\ \_\ \cdots\ \underline{1}}_{n-1 \text{ seq.}}\ \underset{\substack{\uparrow \\ \text{added}}}{1}$$

Then we need to make sure that the $(n-2)$nd entry is not 1: $\underbrace{\_\ \cdots\ \_}_{n-3}\ \overset{\substack{\text{cannot be } 1 \\ \downarrow}}{\underline{0}}\ \underline{1}\ 1$

$\vdots$

So the recurrence relation is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. $\square$