Math 455 : Discrete Structures
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Homework 6 Solutions

§2.3

1) WTS:

$$\binom{n}{k_1,\ldots,k_m} = \binom{n-1}{k_1-1,k_2,\ldots,k_m} + \binom{n-1}{k_1,k_2-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,k_2,\ldots,k_{m-1},k_m-1}$$

By definition, on the LHS we have

$$\binom{n}{k_1,\ldots,k_m} = \frac{n!}{k_1!\cdots k_m!}$$

and on the RHS we have

$$\binom{n-1}{k_1-1,k_2,\ldots,k_m} + \binom{n-1}{k_1,k_2-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,k_2,\ldots,k_{m-1},k_m-1} =$$

$$= \frac{k_1\cdot(n-1)!}{k_1(k_1-1)!k_2!\cdots k_m!} + \frac{k_2\cdot(n-1)!}{k_2\cdot k_1!(k_2-1)!\cdots k_m!} + \cdots + \frac{k_m\cdot(n-1)!}{k_m k_1!k_2!\cdots k_{m-1}!(k_m-1)!}$$

$$= \frac{k_1(n-1)!}{k_1!k_2!\cdots k_m!} + \frac{k_2(n-1)!}{k_1!k_2!\cdots k_m!} + \cdots + \frac{k_m(n-1)!}{k_1!k_2!\cdots k_m!}$$

$$= \frac{k_1(n-1)! + k_2(n-1)! + \cdots + k_m(n-1)!}{k_1!k_2!\cdots k_m!}$$

$$= \frac{(n-1)!(k_1+k_2+\cdots+k_m)}{k_1!k_2!\cdots k_m!}$$

Note: $k_1+k_2+\cdots+k_m = n$

$$= \frac{n!}{k!k_2!\cdots k_m!}$$

$$= \binom{n}{k_1,\ldots,k_m}$$, the LHS.

□

2) Let $a,b,c \in \mathbb{Z}_{>0}$ and let $P(a,b,c)$ denote the number of paths in 3-dim'l space that starts at the origin, ends at (a,b,c), where each "step" is of unit length and is parallel to a coordinate axis.

Consider the polynomial

$$(x+y+z)^{a+b+c} \qquad (*)$$

Each path that we are counting contributes one to the monomial $x^a y^b z^c$ (each variable's power gives the number of steps along that variable's axis).

Then the coefficient of $x^a y^b z^c$ in the expansion of $(*)$ equals $P(a,b,c)$.

§ 2.3

2) (cont.ed)

By (2.22) (Textbook p. 147), this coefficient equals $\binom{a+b+c}{a,\, b,\, c}$.

$$\therefore\ P(a,b,c) = \binom{a+b+c}{a,\, b,\, c}.$$

$\square$

5) Let $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$. WTS:

$$\sum_j \binom{n}{j,\, k,\, n-j-k} = 2^{n-k}\binom{n}{k}$$

$$\sum_j \binom{n}{j,\, k,\, n-j-k} = \sum_j \frac{n!}{j!\,k!\,(n-j-k)!}$$ by the Expansion Identity (2.18), p.145

$$= \sum_j \frac{n!}{k!} \cdot \frac{1}{j!\,(n-j-k)!} \cdot \frac{(n-k)!}{(n-k)!}$$

$$= \sum_j \underbrace{\frac{n!}{k!\,(n-k)!}}_{=\binom{n}{k}} \cdot \underbrace{\frac{(n-k)!}{j!\,(n-k-j)!}}_{=\binom{n-k}{j}}$$

$$= \sum_j \underbrace{\binom{n}{k}}\cdot\binom{n-k}{j}$$

does not depend on j, so we can move it outside of "$\sum_j$"

$$= \binom{n}{k} \cdot \underbrace{\sum_j \binom{n-k}{j}}_{=2^{n-k}}$$ by (2.8), p.140

$$= \binom{n}{k} \cdot 2^{n-k}$$

$\square$

§ 2.3

10) # of 4-letter sequences in BOBO, MISSISSIPPI

vs. distance: 267 mi.

# of 4-letter sequences in SOSO, MISSISSIPPI

We want to fill 4 spots: _ _ _ _

Let X, Y, Z, and W represent distinct letters from the city names. Then we only have five possible "filling patterns". These patterns are really just partitions of the number four:

	filling pattern		Partition of 4
1)	X X X X	(all letters are same)	4
2)	X X X Y	(3 same, one different)	3,1
3)	X X Y Y	(2 pairs of same letters)	2,2
4)	X X Y Z	(1 pair, 2 different)	2,1,1
5)	X Y Z W	(all different letters)	1,1,1,1

Each partition of 4 corresponds to a multinomial coefficient of the form

$$\binom{4}{\text{Partition of } 4}$$

So we have:

	Partition of 4	Corresponding multinomial coefficient
1)	4	$\binom{4}{4} = 1$
2)	3,1	$\binom{4}{3,1} = 4$
3)	2,2	$\binom{4}{2,2} = 6$
4)	2,1,1	$\binom{4}{2,1,1} = 12$
5)	1,1,1,1	$\binom{4}{1,1,1,1} = 24$

Next we need to figure out the possible letter assignments to fill the 4 spots.

§2.3

10) (cont.ed)

- BOBO, MISSISSIPPI has a total of six distinct letters:
  - B (2)
  - O (2)
  - M (1)
  - I (4)
  - S (4)
  - P (2)

filling pattern	# of ways to assign letters to X, Y, Z, W	total possibilities (Just for one filling pattern)
1) XXXX	• 2 (either X = I or X = S) (There are not enough of the other letters to fill four spots.)	$2 \times 1 = 2$
2) XXXY	• 2 choices for X (either X = I or X = S); • 5 choices for Y (since Y can be assigned any letter *except* for the one we chose for X) $\Rightarrow 2 \cdot 5 = 10$ ways	$10 \times 4 = 40$
3) XXYZ	• 5 choices for X (we can have X = B, O, I, S, or P) • $6-1=5$ choices for Y (we may assign *any* of the six letters to Y *except* for what we assigned X). • $6-2=4$ choices for Z (any of the six letters may be assigned to Z *except* for the ones we assigned to X and to Y). $\Rightarrow 5 \cdot 5 \cdot 4 = 100$ ways	$100 \times 12 = 1200$
4) XXYY	• Choose two (different) letters out of the five that occur at least twice $\Rightarrow \binom{5}{2} = 10$ ways	$10 \times 6 = 60$
5) XYZW	• 6 choices for X, since it can be any letter • 5 choices for Y, since " " *except* what we assigned X • 4 choices for Z, since " " *except* what we assigned X and Y, and so on... $\Rightarrow 6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways	$360 \times 24 = 8640$

§2.3

10) (cont.ed)

To find the total number of possibilities (counting all filling patterns), we add the numbers circled in blue on p.4:

Total $= 2+40+1200+60+8640 = 9942$ 4-letter sequences from "BOBO MISSISSIPPI"

- SOSO, MISSISSIPPI has a total of five distinct letters:

S (6)

O (2)

M (1)

I (4)

P (2)

filling pattern	# of ways to assign letters to X, Y, Z, W	total possibilities
1) XXXX	• 2 (either X = S or X = I)	$2\times 1 = 2$
2) XXXY	• 2 choices for X (either X = S or X = I) • 5 choices for Y (since Y can be any letter <u>except</u> the one we assigned to X). $\Rightarrow 2\cdot 5 = 10$ ways	$10\times 4 = 40$
3) XXYZ	• 4 choices for X • $5-1=4$ choices for Y (since $Y\neq X$) • $5-2=3$ choices for Z (since $Z\neq X$ and $Z\neq Y$) $\Rightarrow 4\cdot 4\cdot 3 = 48$ ways	$48\times 12 = 576$
4) XXYY	• Choose any two letters that can be double $\Rightarrow \binom{4}{2} = 6$ ways	$6\times 6 = 36$
5) XYZW	$5\cdot 4\cdot 3\cdot 2 = 120$ ways	$120\times 24 = 2880$

Then the total number of possibilities is the sum of the yellow circled numbers above:

Total $= 2+40+576+36+2880 = 3534$ 4-letter sequences from "SOSO MISSISSIPPI".

(# of 4-letter sequences from "BoBo MISSISSIPPI") − (# of 4-letter sequences from "SOSO MISSISSIPPI") =

$= 9942 - 3534$

$= 6408 > 267$ = distance between Bobo and Soso.

$\square$

§2.3

11) This one is similar to §2.3 Exercise 10.

Lauwiliwilinukunukuoioi

vs. $\Big\}$ fill in 5 spots _ _ _ _ _

humuhumunukunukuapuaa

filling pattern	partition of 5	multinomial coefficient
1) XXXXX	5	$\binom{5}{5} = 1$
2) XXXXY	4,1	$\binom{5}{4,1} = 5$
3) XXXYY	3,2	$\binom{5}{3,2} = 10$
4) XXXYZ	3,1,1	$\binom{5}{3,1,1} = 20$
5) XXYYZ	2,2,1	$\binom{5}{2,2,1} = 30$
6) XXYZW	2,1,1,1	$\binom{5}{2,1,1,1} = 60$
7) XYZWU	1,1,1,1,1	$\binom{5}{1,1,1,1,1} = 120$

- LAUWILIWILINUKUNUKUOIOI has a total of 8 distinct letters:

L(3)
A(1)
U(5)
W(2)
I(6)
N(2)
K(2)
O(2)

(Table on next page)

§2.3

11) (cont.ed)

filling pattern	# of ways to assign letters to X, Y, Z, W, U	total possilalities
1) XXXXX	2	$2 \times 1 = 2$
2) XXXXY	$2 \cdot 7 = 14$	$14 \times 5 = 70$
3) XXXYY	$3 \cdot 6 = 18$	$18 \times 10 = 180$
4) XXXYZ	$3 \cdot 7 \cdot 6 = 126$	$126 \times 20 = 2520$
5) XXYYZ	$\binom{7}{2} \cdot 6 = 21 \cdot 6 = 126$	$126 \times 30 = 3780$
6) XXYZW	$7 \cdot 7 \cdot 6 = 294$	$294 \times 60 = 17{,}640$
7) XYZWU	$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$	$6720 \times 120 = 806{,}400$

Total $= 2 + 70 + 180 + 2520 + 3780 + 17640 + 806400 = 830592$

- HUMUHUMUNUKUNUKUAPUAA has a total of 7 distinct letters:

H (2)
U (9)
M (2)
N (2)
K (2)
A (3)
P (1)

(Table on next page)

§2.3

11) (cont.ed)

filling pattern	# of ways to assign letters to X, Y, Z, W, U	total possibilities
1) XXXXX	1	1
2) XXXXY	6	$6\times5=30$
3) XXXYY	$2\cdot(7-2)=2\cdot5=10$	$10\times10=100$
4) XXXYZ	$2\cdot6\cdot5=60$	$60\times20=1200$
5) XXYYZ	$\binom{6}{2}\cdot5=18\cdot5=90$	$90\times30=2700$
6) XXYZW	$6\cdot6\cdot5\cdot4=720$	$720\times60=43{,}200$
7) XYZWU	$7\cdot6\cdot5\cdot4\cdot3=2520$	$2520\times120=302{,}400$

Total $= 1+30+100+1200+2700+43200+302400 = 349631$

Then

(the # of 5-letter sequences from "HUMUHUMUNUKUNUKUAPUAĀ") $\times 2 = 349631\times2$

$= 699262$

< 830592

$\parallel$

(the # of 5-letter sequences from "LAUWILI WILINUKUNUKUOIOI")

So their claim is correct!

$\square$

§2.5

1) Total # of flags = 50

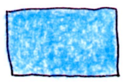
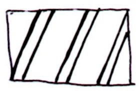
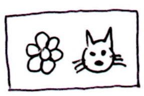
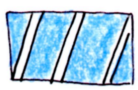
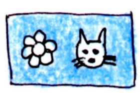
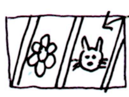
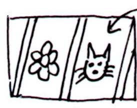
Let B = Blue background

S = Stripes

P = Plant or animal

N = None of the attributes B, S, or P above.

We're given the following information:

Attribute		# of flags with attribute
B		$\lvert B\rvert = 30$
S		$\lvert S\rvert = 12$
P		$\lvert P\rvert = 26$
$B\cap S$		$\lvert B\cap S\rvert = 9$
$B\cap P$		$\lvert B\cap P\rvert = 23$
$S\cap P$	 Color: ?	$\lvert S\cap P\rvert = 3$
$S\cap P\cap B^c$	 Color: not blue	$\lvert S\cap P\cap B^c\rvert = 1$

So $\lvert S\cap P\cap B\rvert = 3-1 = 2$

Then,

$$
\begin{aligned}
N &= 50 - \lvert B\cup S\cup P\rvert \\
&= 50 - \left(\lvert B\rvert + \lvert S\rvert + \lvert P\rvert - \lvert B\cap S\rvert - \lvert B\cap P\rvert - \lvert S\cap P\rvert + \lvert B\cap S\cap P\rvert\right) \\
&= 50 - (30+12+26-9-23-3+2) \\
&= 15
\end{aligned}
$$

$\square$

§2.5

3) Total # of dishes = 35

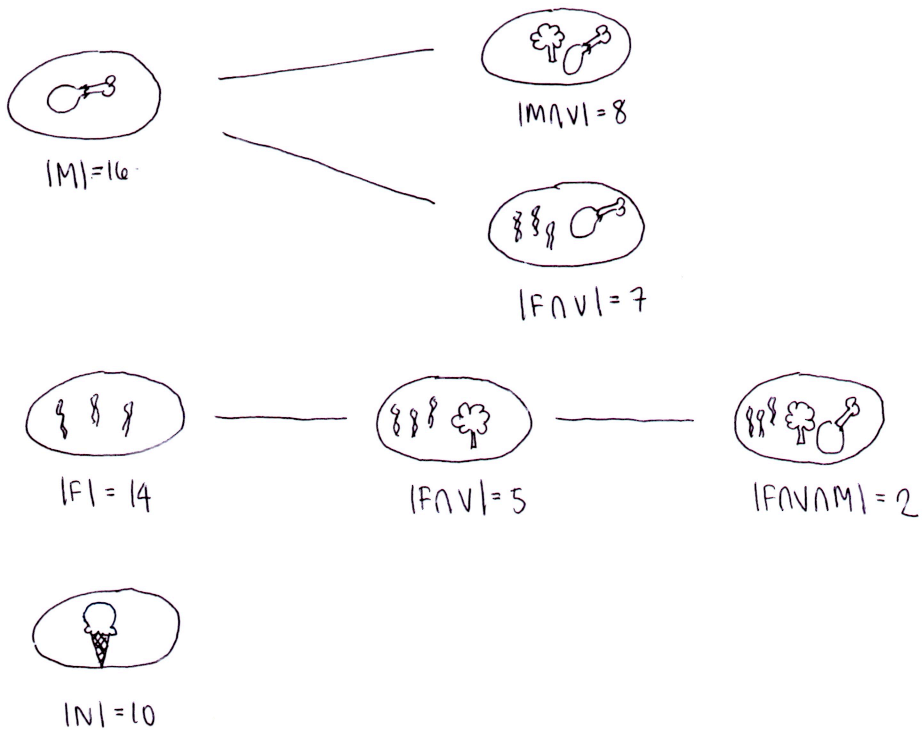
Let M = Meat

V = vegetable

F = Fried

N = None of the attributes M, V, or F

We're given the following information:



Then,

$|N| = 35 - |F \cup M \cup V|$

$10 = 35 - (|F| + |M| + |V| - |F \cap M| - |F \cap V| - |M \cap V| + |F \cap M \cap V|)$

$10 = 35 - (14 + 16 + |V| - 7 - 5 - 8 + 2)$

$\vdots$

So $|V| = 13$.

$\square$

§2.5

6) Total 5-card hands is $\binom{52}{5}$.

Let $a_{\heartsuit}$, $a_{\spadesuit}$, $a_{\clubsuit}$, $a_{\diamondsuit}$ be the property that a given hand contains <u>no</u> cards of that suit.

Then

$$N_{\heartsuit} = \binom{39}{5} = N_{\spadesuit} = N_{\clubsuit} = N_{\diamondsuit}$$

$$N_{\heartsuit\spadesuit} = \binom{26}{5} = N_{\heartsuit\clubsuit} = N_{\heartsuit\diamondsuit} = N_{\clubsuit\diamondsuit} = N_{\diamondsuit\spadesuit} = N_{\clubsuit\diamondsuit}$$

$$N_{\heartsuit\spadesuit\clubsuit} = \binom{13}{5} = N_{\heartsuit\spadesuit\diamondsuit} = N_{\heartsuit\diamondsuit\clubsuit} = N_{\spadesuit\diamondsuit\clubsuit}$$

Let $\heartsuit_0$ = set of all hands with <u>no</u> hearts (and similarly define $\spadesuit_0$, $\clubsuit_0$, and $\diamondsuit_0$). So $|\heartsuit_0| = |\spadesuit_0| = |\clubsuit_0| = |\diamondsuit_0|$.

Then,

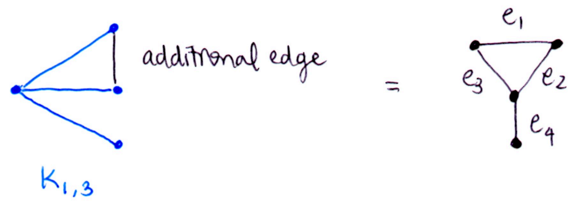
$$\begin{aligned} |\heartsuit_0 \cup \spadesuit_0 \cup \clubsuit_0 \cup \diamondsuit_0| = |\heartsuit_0| + |\spadesuit_0| + |\clubsuit_0| + |\diamondsuit_0| - \\ - N_{\heartsuit} - N_{\spadesuit} - N_{\clubsuit} - N_{\diamondsuit} + \\ + N_{\heartsuit\spadesuit} + N_{\heartsuit\clubsuit} + N_{\heartsuit\diamondsuit} + N_{\diamondsuit\clubsuit} + N_{\diamondsuit\spadesuit} + N_{\clubsuit\diamondsuit} - \\ - N_{\heartsuit\spadesuit\clubsuit} - N_{\heartsuit\spadesuit\diamondsuit} - N_{\spadesuit\diamondsuit\clubsuit} - N_{\heartsuit\diamondsuit\clubsuit} \end{aligned}$$

Then $N_0 = \binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5} = 685464$

$\square$

§2.5

10 (a) Yield sign: $K_{1,3}$ plus one edge



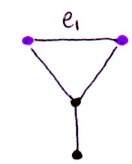
Suppose there are a total of x colors.
In our graph there are 4 vertices.
$\Rightarrow$ there are a total of x^4 possible colorings
$\Rightarrow N = x^4$

☆

Let a_i be the property that edge e_i connects 2 vertices of the same color.
Then,

$N(a_1)$ = # of colorings that satisfy property a_1
= # of colorings in which e_1 connects 2 vertices of the same color

Ex

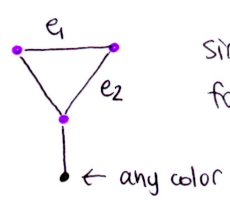


black vertex = any color

and similarly for $N(a_2)$, $N(a_3)$, and $N(a_4)$.

$N(a_1 a_2)$ = # of colorings that satisfy both property a_1 and a_2
= # of colorings in which e_1 connects 2 vertices of the same color, and e_2 connects 2 vertices of the same color

Ex



since e_1 and e_2 share an end vertex, this forces all three vertices to be the same color

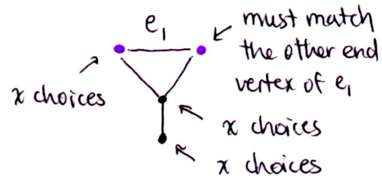
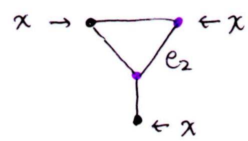
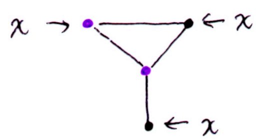
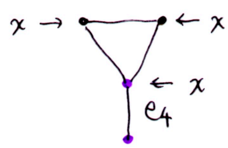
By Thm 2.6 (Principle of Inclusion and Exclusion, p.158):

$$\circledast \quad \underbrace{C_G(x)}_{\substack{\text{chromatic}\\\text{polynomial}\\\text{of } G}} = \underbrace{N}_{\substack{x^4\\\text{(see ☆}\\\text{above)}}} - \sum_i N(a_i) + \sum_{i<j} N(a_i a_j) - \sum_{i<j<k} N(a_i a_j a_k) + \sum_{i<j<k<l} N(a_i a_j a_k a_l)$$

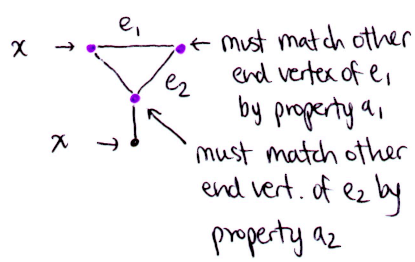
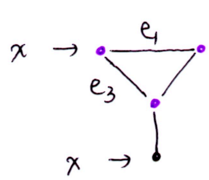
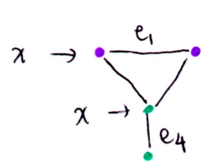
We need to find these

§2.5

10 (a) (cont.ed)

Property a_i	Picture	Value of $N(a_i)$ = # of colorings w/ property a_i
1) a_1		$x \cdot x \cdot x = x^3$
2) a_2		x^3
3) a_3		x^3
4) a_4		x^3

So $\sum_i N(a_i) = N(a_1) + N(a_2) + N(a_3) + N(a_4) = 4x^3$

Properties a_i, a_j	Picture	Value of $N(a_i a_j)$
1) a_1, a_2		$x \cdot x = x^2$
2) a_1, a_3		x^2
3) a_1, a_4		x^2

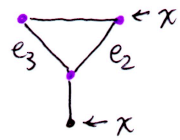
So all vertices in the triangle must be the same color

§2.5

10 (a) (cont.ed)

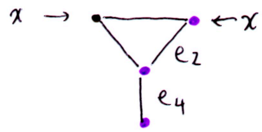
Properties a_i, a_j | Picture | Value of $N(a_i a_j)$ (= # of colorings w/ properties a_i and a_j)

4) a_2, a_3



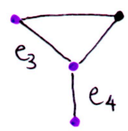
x^2

5) a_2, a_4



x^2

6) a_3, a_4

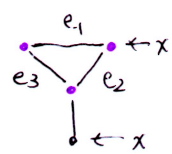


x^2

So $\sum_{i<j} N(a_i a_j) = N(a_1a_2) + N(a_1a_3) + N(a_1a_4) + N(a_2a_3) + N(a_2a_4) + N(a_3a_4) = 6x^2$

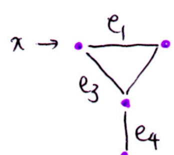
Properties a_i, a_j, a_k | Picture | Value of $N(a_i a_j a_k)$

1) a_1, a_2, a_3



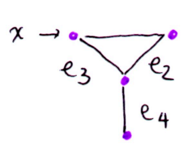
x^2

2) a_1, a_3, a_4



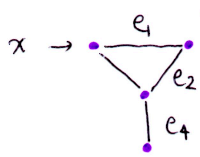
x

3) a_2, a_3, a_4



x

4) a_1, a_2, a_4



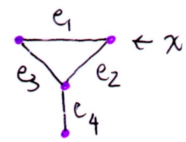
x

So $\sum_{i<j<k} N(a_i a_j a_k) = N(a_1a_2a_3) + N(a_1a_3a_4) + N(a_2a_3a_4) + N(a_1a_2a_4)$

$= x^2 + 3x$

§2.5

10 (a) (cont.ed)

Properties a_i, a_j, a_k, a_l	Picture	Value of $N(a_i a_j a_k a_l)$
1) $N(a_1 a_2 a_3 a_4)$		x

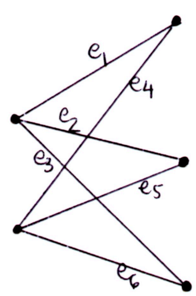
So $\sum_{i<j<k<l} N(a_i a_j a_k a_l) = N(a_1 a_2 a_3 a_4) = x$

Now we can go back to ⊛ on p.11 and fill in the missing parts:

$$C_G(x) = x^4 - 4x^3 + 6x^2 - (x^2 + 3x) + x$$
$$= x^4 - 4x^3 + 5x^2 - 2x$$

□

(b) $K_{2,3}$



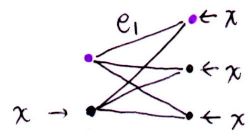
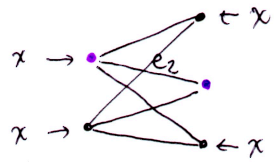
Suppose there are a total of x colors.
$K_{2,3}$ has five vertices.
$\Rightarrow$ there are a total of x^5 possible colorings, so $N = x^5$.

By Thm 2.6,

$$C_G(x) = \underbrace{N}_{x^5} - \underbrace{\sum_i N(a_i)}_{?} + \underbrace{\sum_{i<j} N(a_i a_j)}_{?} - \underbrace{\sum_{i<j<k} N(a_i a_j a_k)}_{?} + \underbrace{\sum_{i<j<k<l} N(a_i a_j a_k a_l)}_{?} - \underbrace{\sum_{i<j<k<l<s} N(a_i a_j a_k a_l a_s)}_{?}$$

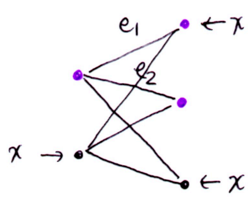
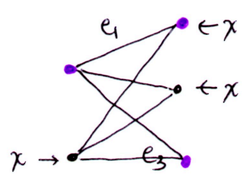
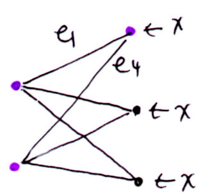
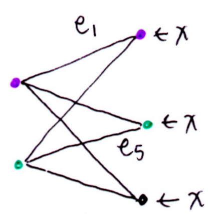
§2.5

10 (b) (cont.ed)

Property a_i	Picture	$N(a_i)$
1) a_1		x^4
2) a_2		x^4
⋮	⋮	⋮

$N(a_i) = x^4$ for each $i = 1, \ldots, 6$.

So $\sum_i N(a_i) = N(a_1) + N(a_2) + \cdots + N(a_6) = 6x^4$

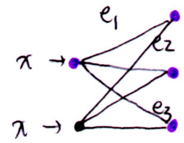
Properties a_i, a_j	Picture	$N(a_i a_j)$
1) a_1, a_2		x^3
2) a_1, a_3		x^3
3) a_1, a_4		x^3
4) a_1, a_5		x^3
⋮	⋮	⋮

$N(a_i a_j) = x^3$ for each $i, j = 1, 2, \ldots, 5$, $i < j$ (there are 15 of these)

So $\sum_{i<j} N(a_i a_j) = 15x^3$

§2.5

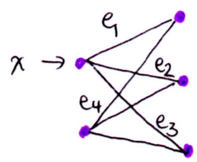
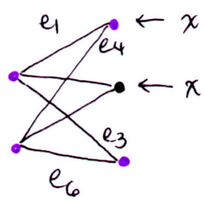
10 (b) (cont.ed)

Properties a_i, a_j, a_k	Picture	$N(a_i a_j a_k)$
1) a_1, a_2, a_3		x^2
⋮		

There are $\underbrace{20}$ of these and $\underbrace{\text{they're all } x^2}$

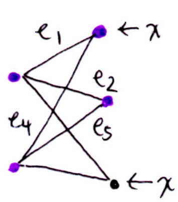
since $\binom{6}{3} = 20$ — since every triple of conditions leaves one black vertex and groups all other vertices as one color, giving $x \cdot x$ colorings

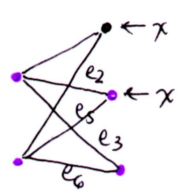
So $\sum_{i<j<k} N(a_i a_j a_k) = 20x^2$.

Properties a_i, a_j, a_k, a_l	Picture	$N(a_i a_j a_k a_l)$
1) a_1, a_2, a_3, a_4		x
2) a_1, a_3, a_4, a_6		x^2
⋮	⋮	⋮

There are 14 of these and they're either x or x^2.
Only 3 are x^2:

$N(a_1 a_2 a_4 a_5) = x^2$ and $N(a_2 a_3 a_5 a_6) = x^2$ and 2) above

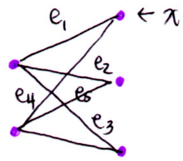




So $\sum_{i<j<k<l} N(a_i a_j a_k a_l) = 11x + 3x^2$

§2.5

10 (b) (cont.ed)

Properties a_i, a_j, a_k, a_l, a_s	Picture	$N(a_i a_j a_k a_l a_s)$
1) a_1, a_2, a_3, a_4, a_5		x

⋮

$N(a_i a_j a_k a_l a_s) = x$ for all i, j, k, l, s, and there are $\binom{6}{5} = 6$ of these.

So $\sum_{i<j<k<l<s} N(a_i a_j a_k a_l a_s) = 6x$

Then,

$$C_G(x) = x^5 - 6x^4 + 15x^3 - 20x^2 + (11x + 3x^2) - 6x$$
$$= x^5 - 6x^4 + 15x^3 - 17x^2 + 5x$$

□

11) For $n = 12$, the probability is $0.3678794413212\ldots$
and $n = 120$, " " $0.3678794411714\ldots$
Difference $\approx 10^{-10}$ (very small)

□