Math 455: Discrete Structures

Prof. Paul Gunnells

TA: Jennifer Li

Homework 5 Solutions

§2.1

1 (a)

Total # of choices: $53 \cdot 63 \cdot 63 \cdot 63 \cdot 63 = 53 \cdot 63^4$

Reason: { Characters allowed: _, a, ..., z, A, ..., Z ; Total # of choices: $1+26+26=53$
" " : _, a, ..., z, A, ..., Z, 0, ..., 9 ; " " : $1+26+26+10=63$

(b) 1 character: $53 = 53$

OR

2 char.s: $53 \cdot 63 = 53 \cdot 63$

OR

3 char.s: $53 \cdot 63 \cdot 63 = 53 \cdot 63^2$

OR

4 char.s: $53 \cdot 63 \cdot 63 \cdot 63 = 53 \cdot 63^3$

OR

5 char.s: $53 \cdot 63 \cdot 63 \cdot 63 \cdot 63 = 53 \cdot 63^4$

$53 + 53 \cdot 63 + 53 \cdot 63^2 + 53 \cdot 63^3 + 53 \cdot 63^4$ = total # of choices □

(c) Max. length is 5

Case (i) Suppose the length is even. Then the possibilities are:

2 char.s: ★ | ★
53, mirror — forced to be whatever ★ we chose in the blue spot, since this is a palindrome.

OR

4 char.s: ★ △ | △ ★
53 63, mirror — both forced (cannot choose these)

OR

Case (ii) Suppose the length is odd. Then the possibilities are:

1 char.: 53

OR

3 char.s: ★ _ | ★
53, 63 — not affected by mirror; mirror; forced

OR

5 char.s: ★ △ _ | △ ★
53 63 63, mirror

Then the total # of choices is: $53 + 53 \cdot 63 + 53 + 53 \cdot 63 + 53 \cdot 63^2$ □

§2.1

2 (a) We have 11 slots and we want exactly 3 vowels (either $\underbrace{A, E, I, O, \text{ or } U}_{5 \text{ vowels}}$)

There are $\binom{11}{3}$ ways to pick which slots the vowels will go in;
In each of those spots, we have 5 choices of vowels (so together that's $5\cdot5\cdot5$ choices);
To ensure that those are the only vowels in the sequence, we want to place nonvowels in the remaining $11-3=8$ slots. There are a total of $26-5=21$ nonvowels. So this gives 21^8 choices.
Then the total # of possibilities is $\binom{11}{3}\cdot 5^3\cdot 21^8$.

□

(b) Let's find first the total number of possibilities to get:
an 11-letter sequence with exactly three vowels without repetition. ⊛

* [Then the total number of 11-letter sequences with exactly three vowels having at least one repetition is:

[Total # from (a)] − [Total # of type ⊛]

There are $5\cdot4\cdot3$ ways to choose three different vowels.
There are

$$\frac{21!}{(21-8)!}$$

ways to choose eight different nonvowels.
Then there are

$$\binom{11}{3}\cdot 5\cdot4\cdot3\cdot\frac{21!}{(21-8)!}$$

ways to get sequences of type ⊛
Then the total ways to get sequences of type * is

$$\underbrace{\binom{11}{3}\cdot 5^3\cdot 21^8}_{\text{from (a)}} - \underbrace{\binom{11}{3}\cdot 5\cdot4\cdot3\cdot\frac{21!}{(21-8)!}}_{\text{type } ⊛}$$

□

§2.1

6) The exam format looks like :

Part I

There are 10 of these types, or a total of 16^{10} ways to complete Part I

Ques 1 : Multiple choice

(a) Can choose or not choose $\rightsquigarrow$ 2 choices
(b) " " $\rightsquigarrow$ " "
(c) " " $\rightsquigarrow$ " "
(d) " " $\rightsquigarrow$ " "

total : 2^4 choices $= 16$

⋮

Ques 10 : " "

Part I Choose one of the following subparts :

Since we can choose to do either subpart 1 OR subpart 2, there are $2^8 + 10^7$ ways to complete Part II

Subpart 1

Ques 1 : True or False $\rightsquigarrow$ 2 choices

⋮

Ques 8 : " " $\rightsquigarrow$ " "

total : 2^8 Choices

Subpart 2 Choose the proper def. of each term

7 terms: Term 1, Term 2, ⋮, Term 7

10 Def.s: Def 1, Def 2, ⋮, Def 10

total : 10^7 choices

$\therefore$ Total ways to complete exam $= 16^{10} \cdot (2^8 + 10^7)$.

$\square$

8 (a) Ordering the cards from highest to lowest :

$\underline{\quad}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}$

↑ highest : A, K, Q, J, 10, 9, ..., 6, 5 (total : 10 choices)

determined by highest card ; since there are four suits {♥, ♠, ♣, ♦} there are a total of $\binom{4}{1}$ choices for each of the five slots.

Then we have $10 \cdot \binom{4}{1}^5$ choices of filling these blanks.

But we need to exclude the case where we have a straight flush (all cards are of the same suit). There are a total of 40 of these (See (c)), so we have

$$10 \cdot \binom{4}{1}^5 - 40 = (10 \cdot 4^5) - 40 \text{ possibilities.}$$

§2.1

8 (cont.ed)

(b) All five cards are of the same suit (excluding straight flushes)

$\underbrace{\heartsuit\ \spadesuit\ \clubsuit\ \diamondsuit}_{\text{4 choices for suits}} \rightsquigarrow \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}$

then fill these blanks with cards of the suit we chose; there are $\binom{13}{5}$ possible five-card hands of the same suit

Total: $4 \cdot \binom{13}{5} - \underbrace{40}$ possibilities

we want to exclude straight flushes, and there are a total of 40 of these (see (c))

(c) The values of cards are in consecutive order <u>and</u> it is a flush

$\underbrace{\heartsuit\ \spadesuit\ \clubsuit\ \diamondsuit}_{\text{4 choices for suits}} \rightsquigarrow \underline{\quad}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}$

(suppose we order the cards in decreasing order of their value)

Choosing card #1 determines all of these.

Card #1 can be A, K, Q, J, 10, 9, ..., or 5 (10 choices)

Total: $4 \cdot 10$ possibilities

(d) We want 5 cards such that

$\underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}\ \underline{\quad}$

Four cards are of the same value (or rank)

The possible values are: 1, 2, 3, ..., 12, 13 (13 choices)

There are exactly 4 cards of each rank; so

$\underline{\quad}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}}\ \underline{\quad}$

we don't care what card this is; so there are $52-4=48$ possibilities for the 5th card

once a rank (13 choices) is chosen, the next three cards are determined.

Total: $4 \cdot 48$ possibilities.

§2.1

8) (cont-ed)

(e) Two matching (distinct) pairs

This means that we want:

Card 1	Card 2	Card 3	Card 4	Card 5
Value A	Value A	Value B	Value B	Value C
♥ ♠ ♣ ♦		♥ ♠ ♣ ♦		♥ ♠ ♣ ♦

- First choose Value A and Value B (say, card 1 and card 3) out of the 13 total possible values. There are a total of $\binom{13}{2}$ ways to do this (notice that this guarantees that Value A and Value B will be distinct).
- Then Card 2 and card 4 are determined.
- Now the two Value A cards have a suit (4 possibilities) but they cannot have the same suit (because of each value, there is only one of each suit of that value; for instance there is only one heart, one spades, one clubs, and one diamond of the card value 12). Then for the value A cards, there are $\binom{4}{2}$ possible choices for the suits of card 1 and card 2. The same is true for the Value B cards: $\binom{4}{2}$ possibilities.
- So far we have the possibilities for cards 1-4. There is just card 5 left. We just need to ensure that Value C is different from Value A and Value B. So there are $\binom{13-2}{1} = \binom{11}{1}$ choices for Value C. There are $\binom{4}{1}$ choices for the suit of card 5.

Otherwise we would have one pair and a three-of-a-kind; this is called a Full House, which is a better hand than two matching pairs!

$\therefore$ the total # of possibilities is $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{11}{1} \cdot \binom{4}{1}$

(f) Exactly one matching pair

card 1	card 2	card 3	card 4	card 5
Value A	Value A	Value B	Value C	Value D
♥ ♠ ♣ ♦				

§2.1

8) (cont.ed)

(f) • For Value A there are $\binom{13}{1}$ choices; this determines card 1 and card 2. There are $\binom{4}{2}$ choices for their suits.

• Cards 3-5 must have:

- different values from Value A
- different values from each other

$\rightsquigarrow \binom{13-1}{3} = \binom{12}{3}$ choices

- They can each be of any of the 4 suits, so we have $\binom{4}{1}$ choices for suits for each card.

$\therefore$ total # of possibilities is $\binom{13}{1}\cdot\binom{4}{2}\cdot\binom{12}{3}\cdot\binom{4}{1}^3$.

(g) At least one card from each suit:

• Since this is a 5-card hand and there are 4 suits, we know that one suit must appear twice. There are $\binom{4}{1}$ possibilities for this repeated suit.

Card 1	Card 2	Card 3	Card 4	Card 5
Suit 1	Suit 1	Suit 2	Suit 3	Suit 4
A, 2, ..., K		A, 2, ..., K	A, 2, ..., K	A, 2, ..., K

• Each suit has exactly one value of the 13 values, so there are $\binom{13}{2}$ choices for the values of card 1 and card 2.

• The remaining cards' suits are determined; there are $\binom{13}{1}$ choices for the values of each of card 3, card 4, and card 5.

$\therefore$ total # of possibilities is $\binom{4}{1}\cdot\binom{13}{2}\cdot\binom{13}{1}^3$.

(h) At least one card from each suit but no matching values.

This is similar to (g), except the values of cards 3-5 are chosen differently.

• Value of card 3 has $\binom{13-2}{1} = \binom{11}{1}$ possibilities (it must be different from the values of cards 1 and 2.

• Value of card 4 has $\binom{13-3}{1} = \binom{10}{1}$ possibilities (it's different from values of cards 1-3)

• Value of card 5 has $\binom{13-4}{1} = \binom{9}{1}$ possibilities.

Then the total # of possibilities is $\underbrace{\binom{4}{1}\cdot\binom{13}{2}}_{\text{cards 1 and 2}}\cdot\binom{11}{1}\binom{10}{1}\binom{9}{1}$

§2.1

8) (Cont.ed)

(i) Three cards of one suit and the other two of another suit.

card 1	card 2	card 3	card 4	card 5
Suit 1	Suit 1	Suit 1	Suit 2	Suit 2
A, 2, ..., K			A, 2, ..., K	

- First let's choose Suit 1 and Suit 2. There are $\binom{4}{2}$ ways to do this.
- Now we need to choose the "triple" suit. There are $\binom{2}{1}$ ways to do this.
  Note This is important, because otherwise we would not be distinguishing between 3 spades + 2 hearts, and 2 spades + 3 hearts.
  (Once the triple suit is chosen, the remaining suit is automatically the double suit).
- Finally we need to choose the values of the suits. The three Suit 1 card values can be chosen from a single pool of values $\{A, 2, \ldots, K\}$ and the two Suit 2 card values are chosen from another pool $\{A, 2, \ldots, K\}$, giving $\binom{13}{3} \cdot \binom{13}{2}$ choices.

$\therefore$ the total # of possibilities is $\binom{4}{2} \cdot \binom{2}{1} \cdot \binom{13}{3} \cdot \binom{13}{2}$.

$\square$

the set of all positive integers

11) Suppose that $N \in \mathbb{Z}^+$ factors as

$$N = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m},$$

where $p_1, p_2, \ldots, p_m$ are distinct primes and $n_1, n_2, \ldots, n_m \in \mathbb{Z}^+$

Given a divisor:

p_1 can appear between 1 and n_1 times $\rightsquigarrow$ n_1 total

p_2 " " 1 and n_2 times $\rightsquigarrow$ n_2 total

$\vdots$

p_m " " 1 and n_m times $\rightsquigarrow$ n_m total

Then total # of choices is $n_1 \cdot n_2 \cdot \ldots \cdot n_m$.

$\square$

§2.2

2) WTS: $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

On the LHS (left-hand side) we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n\cdot(n-1)!}{k\cdot(k-1)!(n-k)!}$$

$$= \frac{n}{k}\cdot\frac{(n-1)!}{(k-1)!(n-k)!}$$

On the RHS we have

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n}{k}\cdot\frac{(n-1)!}{(k-1)!\cdot(n-1-(k-1))!}$$

$$= \frac{n}{k}\cdot\frac{(n-1)!}{(k-1)!(n-k)!}$$

So LHS = RHS.

$\square$

3) Let n, k be nonnegative integers and m an integer with $m \leq n$. Then WTS

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{k!(n-k)!}\cdot\frac{k!}{m!(k-m)!}$$

$$= \frac{n!}{\cancel{k!}(n-k)!}\cdot\frac{\cancel{k!}}{m!(k-m)!}\cdot\underbrace{\frac{(n-m)!}{(n-m)!}}_{1}$$

$$= \frac{n!}{m!(n-m)!}\cdot\frac{(n-m)!}{(k-m)!(n-k)!}$$

$$= \binom{n}{m}\cdot\binom{n-m}{k-m}$$

$\square$

5) Let m, n be nonnegative integers. WTS:

$$\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$$

§2.2

5) (cont.ed)

We can prove this by induction on n:

Base Case ($n=1$)

$$\sum_{k=0}^{1} \binom{m+k}{k} = \binom{m+0}{0} + \binom{m+1}{1} = m+2 = \binom{m+1+1}{1}$$

Inductive step

Suppose that

$$\sum_{k=0}^{t} \binom{m+k}{k} = \binom{m+t+1}{t} \quad ⊛$$

WTS the equality is also true for $t+1$; i.e., that $\sum_{k=0}^{t+1} \binom{m+k}{k} = \binom{m+t+2}{n}$

We can write ⊛ as:

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+t}{t} = \binom{m+t+1}{t}$$

Adding $\binom{m+t+1}{t+1}$ to each side of the equation above, we have

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+t}{t} + \binom{m+t+1}{t+1} = \binom{m+t+1}{t} + \binom{m+t+1}{t+1}$$

$$= \binom{m+t+2}{t+1} \quad \text{by (2.6) on p.138 (Additive Identity)}$$

$$\Rightarrow \sum_{k=0}^{t+1} \binom{m+k}{k} = \binom{m+(t+1)+1}{n},$$ so the equality ⊛ holds for $t+1$.

By induction ⊛ holds for all integers t, and therefore

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n} \quad \text{for any } n \in \mathbb{Z}^+.$$

□

§2.2

6) Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$. WTS:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

Using the definition, we can write the LHS as

$$\binom{n-1}{k-1}\cdot\binom{n}{k+1}\cdot\binom{n+1}{k} = \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}\cdot\frac{n!}{(k+1)!\,(n-k-1)!}\cdot\frac{(n+1)!}{k!\,(n+1-k)!}$$

and the RHS as

$$\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \frac{(n-1)!}{k!\,(n-1-k)!}\cdot\frac{n!}{(k-1)!\,(n-k+1)!}\cdot\frac{(n+1)!}{(k+1)!\,(n+1-k-1)!}$$

proving the equality holds.

$\square$

8) Prove

(i)

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n}\binom{n}{k} x^{\underline{k}}\, y^{\underline{n-k}} \quad , \text{ and}$$

(ii)

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n}\binom{n}{k} x^{\overline{k}}\, y^{\overline{n-k}}$$

We can prove these by induction on n.

For (i):

Base case $(n=1)$

$$\sum_{k=0}^{1}\binom{1}{k} x^{\underline{k}}\, y^{\underline{1-k}} = \binom{1}{0} x^{\underline{0}}\, y^{\underline{1-0}} + \binom{1}{1} x^{\underline{1}}\, y^{\underline{0}} = (x+y)^{\underline{1}}$$

Inductive step

Suppose the equality holds for t; WTS it holds for $t+1$. That is,

$$(x+y)^{\underline{t+1}} = \sum_{k=0}^{t+1}\binom{t+1}{k} x^{\underline{k}}\, y^{\underline{(t+1)-k}}$$

§2.2

8 (cont.ed)

We can write:

$$\sum_{k=0}^{t+1} \binom{t+1}{k} x^k y^{t+1-k} = \sum_{k=0}^{t+1} \left(\binom{t}{k} + \binom{t}{k-1} \right) x^k y^{t+1-k}$$

$$= \sum_{k=0}^{t} \binom{t}{k} x^k y^{t+1-k} + \sum_{k=0}^{t} \binom{t}{k} x^{k+1} y^{t-k}$$

$$= \sum_{k=0}^{t} \binom{t}{k} \left(x^k y^{t+1-k} + x^{k+1} y^{t-k} \right)$$

$$= \sum_{k=0}^{t} \binom{t}{k} x^k y^{t-k} \left((y-t+k) + (x-k) \right)$$

$$= \sum_{k=0}^{t} \binom{t}{k} x^k y^{t-k} (y-t+x)$$

$$= (x+y-t)(x+y)^{t} \qquad \text{since } \sum_{k=0}^{t} \binom{t}{k} x^k y^{t-k} = (x+y)^t \text{ by inductive hypothesis}$$

$$= (x+y)^{t+1}$$

(ii) is similar (proof by induction). □