

Homework 5 Solutions

§2.1

1 (a)

Total # of choices:  $53 \cdot 63 \cdot 63 \cdot 63 \cdot 63 = 53 \cdot 63^4$

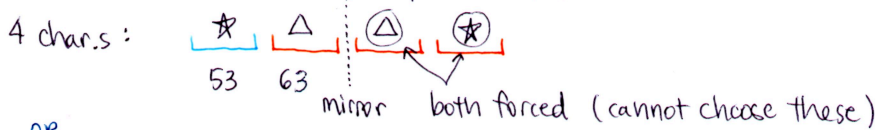
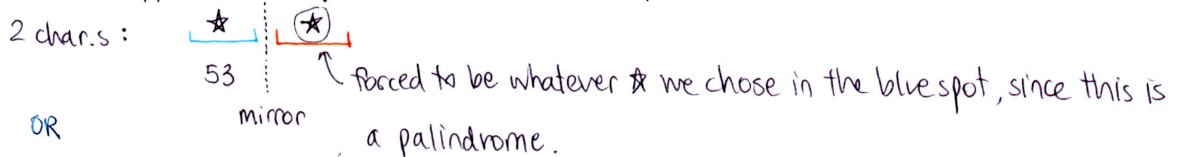
Reason: { Characters allowed:  $\_$ , a, ..., z, A, ..., Z ; Total # of choices:  $1+26+26 = 53$   
 " " :  $\_$ , a, ..., z, A, ..., Z, 0, ..., 9 ; " " :  $1+26+26+10 = 63$

(b) 1 character :

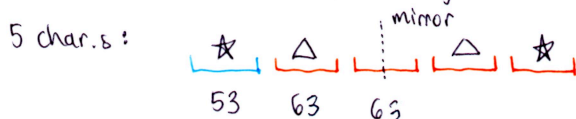
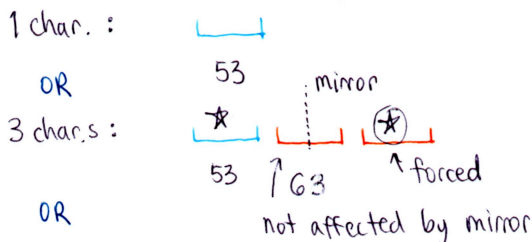
OR		=	53	} = total # of choices
2 char.s :		=	53 + 53 \cdot 63	
OR		=	53 \cdot 63	
3 char.s :		=	53 + 53 \cdot 63 + 53 \cdot 63^2	
OR		=	53 \cdot 63^2	
4 char.s :		=	53 + 53 \cdot 63 + 53 \cdot 63^2 + 53 \cdot 63^3	
OR		=	53 \cdot 63^3	
5 char.s :		=	53 + 53 \cdot 63 + 53 \cdot 63^2 + 53 \cdot 63^3 + 53 \cdot 63^4	
OR		=	53 \cdot 63^4	

(c) Max. length is 5

Case (i) Suppose the length is even. Then the possibilities are:



Case (ii) Suppose the length is odd. Then the possibilities are:



Then the total # of choices is :  $53 + 53 \cdot 63 + 53 + 53 \cdot 63 + 53 \cdot 63^2$

## §2.1

2 (a) We have 11 slots and we want exactly 3 vowels (either A, E, I, O, or U)

5 vowels

There are  $\binom{11}{3}$  ways to pick which slots the vowels will go in;

In each of those spots, we have 5 choices of vowels (so together that's  $5 \cdot 5 \cdot 5$  choices);

To ensure that those are the only vowels in the sequence, we want to place

nonvowels in the remaining  $11-3=8$  slots. There are a total of  $26-5=21$  nonvowels. So this gives  $21^8$  choices.

Then the total # of possibilities is  $\binom{11}{3} \cdot 5^3 \cdot 21^8$ .

□

(b) Let's find first the total number of possibilities to get:

an 11-letter sequence with exactly three vowels without repetition. \*

\* [ Then the total number of 11-letter sequences with exactly three vowels having at least one repetition is:

[ Total # from (a) ] - [ Total # of type \*

There are  $5 \cdot 4 \cdot 3$  ways to choose three different vowels.

There are

$$\frac{21!}{(21-8)!}$$

ways to choose eight different nonvowels.

Then there are

$$\binom{11}{3} \cdot 5 \cdot 4 \cdot 3 \cdot \frac{21!}{(21-8)!}$$

ways to get sequences of type \*

Then the total ways to get sequences of type \* is

$$\underbrace{\binom{11}{3} \cdot 5^3 \cdot 21^8}_{\text{from (a)}} - \underbrace{\binom{11}{3} \cdot 5 \cdot 4 \cdot 3 \cdot \frac{21!}{(21-8)!}}_{\text{type *}}$$

□

§2.1

6) The exam format looks like :

Part I

Ques 1 : Multiple choice

- (a) Can choose or not choose  $\rightsquigarrow$  2 choices
- (b) " " " "  $\rightsquigarrow$  " "
- (c) " " " "  $\rightsquigarrow$  " "
- (d) " " " "  $\rightsquigarrow$  " "

} total:  $2^4$  choices  
" 16

There are 10 of these types, or a total of  $16^{10}$  ways to complete Part I

Ques 10 : " "

Part I Choose one of the following subparts :

Subpart 1

Ques 1 : True or False  $\rightsquigarrow$  2 choices

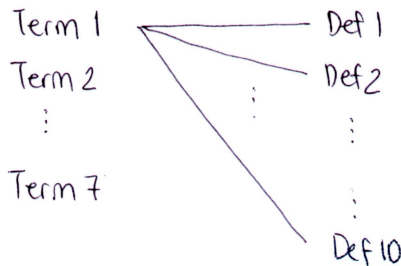
⋮

Ques 8 : " " " "  $\rightsquigarrow$  " "

} total:  $2^8$  choices

Since we can choose to do either subpart 1 OR subpart 2, there are  $2^8 + 10^7$  ways to complete Part II

Subpart 2 Choose the proper def. of each term



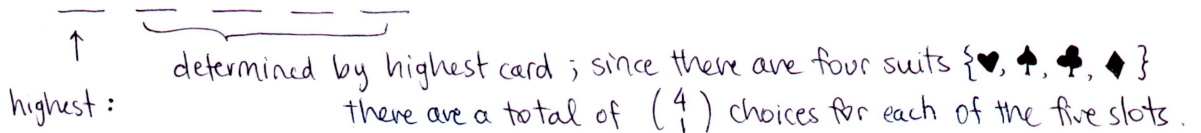
} 10 Def.s

} total:  $10^7$  choices

$\therefore$  Total ways to complete exam =  $16^{10} \cdot (2^8 + 10^7)$ .

□

8 (a) Ordering the cards from highest to lowest :



A, K, Q, J, 10, 9, ..., 6, 5  
(total: 10 choices)

Then we have  $10 \cdot \binom{4}{1}^5$  choices of filling these blanks.

But we need to exclude the case where we have a straight flush (all cards are of the same suit). There are a total of 40 of these (See (c)), so we have

$10 \cdot \binom{4}{1}^5 - 40 = (10 \cdot 4^5) - 40$  possibilities.

§2.1

8 (cont-ed)

(b) All five cards are of the same suit (excluding straight flushes)



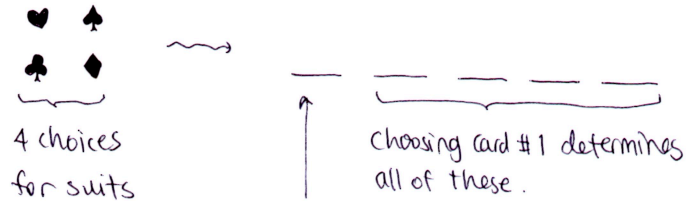
4 choices for suits

then fill these blanks with cards of the suit we chose; there are  $\binom{13}{5}$  possible five-card hands of the same suit

Total:  $4 \cdot \binom{13}{5} = 40$  possibilities

we want to exclude straight flushes, and there are a total of 40 of these (see (c))

(c) The values of cards are in consecutive order and it is a flush



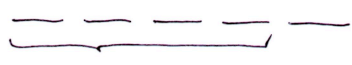
4 choices for suits

(suppose we order the cards in decreasing order of their value)

Card #1 can be A, K, Q, J, 10, 9, ..., or 5 (10 choices)

Total:  $4 \cdot 10$  possibilities

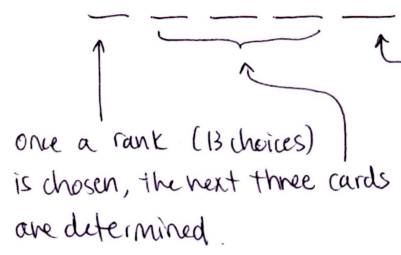
(d) We want 5 cards such that



Four cards are of the same value (or rank)

The possible values are: 1, 2, 3, ..., 12, 13 (13 choices)

There are exactly 4 cards of each rank; so



we don't care what card this is; so there are  $52 - 4 = 48$  possibilities for the 5th card

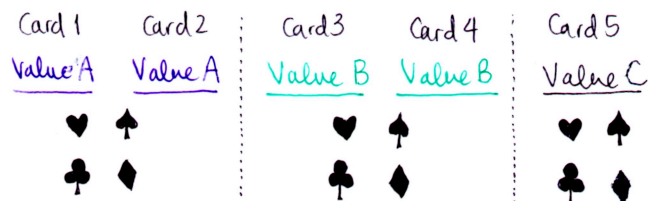
Total:  $4 \cdot 48$  possibilities.

## §2.1

8) (cont'd)

(e) Two matching (distinct) pairs

This means that we want:

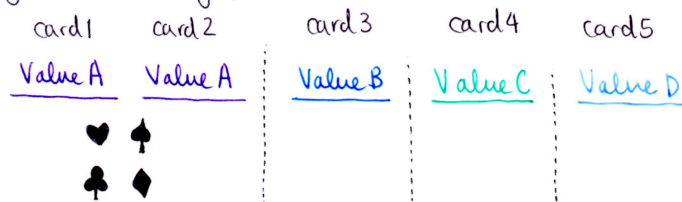


- First choose Value A and Value B (say, card 1 and card 3) out of the 13 total possible values. There are a total of  $\binom{13}{2}$  ways to do this (notice that this guarantees that Value A and Value B will be distinct).
- Then Card 2 and card 4 are determined.
- Now the two Value A cards have a suit (4 possibilities) but they cannot have the same suit (because of each value, there is only one of each suit of that value; for instance there is only one heart, one spades, one clob, and one diamond of the card value 12). Then for the value A cards, there are  $\binom{4}{2}$  possible choices for the suits of card 1 and card 2. The same is true for the Value B cards:  $\binom{4}{2}$  possibilities.
- So far we have the possibilities for cards 1-4. There is just card 5 left. We just need to ensure that Value C is different from Value A and Value B.  $\hookrightarrow$   
So there are  $\binom{13-2}{1} = \binom{11}{1}$  choices for Value C.  
There are  $\binom{4}{1}$  choices for the suit of card 5.

otherwise we would have one pair and a three-of-a-kind; this is called a Full House, which is a better hand than two matching pairs!

$$\therefore \text{the total \# of possibilities is } \binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{11}{1} \cdot \binom{4}{1}$$

(f) Exactly one matching pair



§2.1

8) (cont.ed)

(f) • For Value A there are  $\binom{13}{1}$  choices; this determines card 1 and card 2.  
 There are  $\binom{4}{2}$  choices for their suits.

- Cards 3-5 must have :
  - different values from Value A  $\xrightarrow{\sim} \binom{13-1}{1} = \binom{12}{1}$  choices
  - different values from each other  $\xrightarrow{\sim} \binom{3}{1} = \binom{12}{3}$  choices
  - They can each be of any of the 4 suits, so we have  $\binom{4}{1}$  choices for suits for each card.

∴ total # of possibilities is  $\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1}^3$ .

(g) At least one card from each suit :

- Since this is a 5-card hand and there are 4 suits, we know that one suit must appear twice. There are  $\binom{4}{1}$  possibilities for this repeated suit.

Card 1	Card 2	Card 3	Card 4	Card 5
<u>Suit 1</u>	<u>Suit 1</u>	<u>Suit 2</u>	<u>Suit 3</u>	<u>Suit 4</u>
A, 2, ..., K	A, 2, ..., K	A, 2, ..., K	A, 2, ..., K	A, 2, ..., K

- Each suit has exactly one value of the 13 values, so there are  $\binom{13}{2}$  choices for the values of card 1 and card 2.
- The remaining cards' suits are determined; there are  $\binom{13}{1}$  choices for the values of each of card 3, card 4, and card 5.

∴ total # of possibilities is  $\binom{4}{1} \cdot \binom{13}{2} \cdot \binom{13}{1}^3$ .

(h) At least one card from each suit but no matching values.

This is similar to (g), except the values of cards 3-5 are chosen differently.

- Value of card 3 has  $\binom{13-2}{1} = \binom{11}{1}$  possibilities (it must be different from the values of cards 1 and 2).
- Value of card 4 has  $\binom{13-3}{1} = \binom{10}{1}$  possibilities (it's different from values of cards 1-3)
- Value of card 5 has  $\binom{13-4}{1} = \binom{9}{1}$  possibilities.

Then the total # of possibilities is  $\underbrace{\binom{4}{1} \cdot \binom{13}{2}}_{\text{cards 1 and 2}} \cdot \binom{11}{1} \cdot \binom{10}{1} \cdot \binom{9}{1}$

## §2.1

8) (cont.ed)

(i) Three cards of one suit and the other two of another suit.

card 1	card 2	card 3		card 4	card 5
<u>Suit 1</u>	<u>Suit 1</u>	<u>Suit 1</u>		<u>Suit 2</u>	<u>Suit 2</u>
A, 2, ..., K				A, 2, ..., K	

- First let's choose Suit 1 and Suit 2. There are  $\binom{4}{2}$  ways to do this.
- Now we need to choose the "triple" suit. There are  $\binom{2}{1}$  ways to do this.  
Note This is important, because otherwise we would not be distinguishing between 3 spades + 2 hearts, and 2 spades + 3 hearts.  
 (Once the triple suit is chosen, the remaining suit is automatically the double suit).
- Finally we need to choose the values of the suits. The three Suit 1 card values can be chosen from a single pool of values  $\{A, 2, \dots, K\}$  and the two Suit 2 card values are chosen from another pool  $\{A, 2, \dots, K\}$ , giving  $\binom{13}{3} \cdot \binom{13}{2}$  choices.

$\therefore$  the total # of possibilities is  $\binom{4}{2} \cdot \binom{2}{1} \cdot \binom{13}{3} \cdot \binom{13}{2}$ .

□

11) Suppose that  $N \in \mathbb{Z}^+$  factors as

$$N = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m},$$

where  $p_1, p_2, \dots, p_m$  are distinct primes and  $n_1, n_2, \dots, n_m \in \mathbb{Z}^+$

Given a divisor:

$p_1$  can appear between 1 and  $n_1$  times  $\rightsquigarrow n_1$  total

$p_2$  " " 1 and  $n_2$  times  $\rightsquigarrow n_2$  total

⋮

$p_m$  " " 1 and  $n_m$  times  $\rightsquigarrow n_m$  total

Then total # of choices is  $n_1 \cdot n_2 \cdot \dots \cdot n_m$ .

□

## §2.2

$$2) \text{ WTS: } \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

On the LHS (left-hand side) we have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n \cdot (n-1)!}{k \cdot (k-1)! \cdot (n-k)!} \\ &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)! \cdot (n-k)!} \end{aligned}$$

On the RHS we have

$$\begin{aligned} \frac{n}{k} \binom{n-1}{k-1} &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)! \cdot (n-1-(k-1))!} \\ &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)! \cdot (n-k)!} \end{aligned}$$

So LHS = RHS. □

3) Let  $n, k$  be nonnegative integers and  $m$  an integer with  $m \leq n$ . Then WTS

$$\begin{aligned} \binom{n}{k} \binom{k}{m} &= \binom{n}{m} \binom{n-m}{k-m} \\ \binom{n}{k} \binom{k}{m} &= \frac{n!}{k!(n-k)!} \cdot \frac{k!}{m!(k-m)!} \\ &= \frac{n!}{\cancel{k!} (n-k)!} \cdot \frac{\cancel{k!}}{m!(k-m)!} \cdot \frac{(n-m)!}{\underbrace{(n-m)!}_1} \\ &= \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} \\ &= \binom{n}{m} \cdot \binom{n-m}{k-m} \end{aligned} \quad \square$$

5) Let  $m, n$  be nonnegative integers. WTS:

$$\sum_{k=0}^n \binom{m+k}{k} = \binom{m+n+1}{n}$$



§2.2

5) (cont.ed)

We can prove this by induction on  $n$ :

Base Case ( $n=1$ )

$$\sum_{k=0}^1 \binom{m+k}{k} = \binom{m+0}{0} + \binom{m+1}{1} = m+2 = \binom{m+1+1}{1}$$

Inductive step

Suppose that

$$\sum_{k=0}^t \binom{m+k}{k} = \binom{m+t+1}{t} \quad (\star)$$

WTS the equality is also true for  $t+1$ ; i.e., that  $\sum_{k=0}^{t+1} \binom{m+k}{k} = \binom{m+t+2}{t+1}$

We can write  $(\star)$  as:

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \dots + \binom{m+t}{t} = \binom{m+t+1}{t}$$

Adding  $\binom{m+t+1}{t+1}$  to each side of the equation above, we have

$$\begin{aligned} \binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \dots + \binom{m+t}{t} + \binom{m+t+1}{t+1} &= \binom{m+t+1}{t} + \binom{m+t+1}{t+1} \\ &= \binom{m+t+2}{t+1} \quad \text{by (2.6) on p.138} \\ &\quad \text{(Additive Identity)} \end{aligned}$$

$\Rightarrow \sum_{k=0}^{t+1} \binom{m+k}{k} = \binom{m+(t+1)+1}{t+1}$ , so the equality  $(\star)$  holds for  $t+1$ .

By induction  $(\star)$  holds for all integers  $t$ , and therefore

$$\sum_{k=0}^n \binom{m+k}{k} = \binom{m+n+1}{n} \quad \text{for any } n \in \mathbb{Z}^+.$$

□

§2.2

6) Let  $n \in \mathbb{Z}^+$  and  $k \in \mathbb{Z}$ . WTS:

$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1}$$

Using the definition, we can write the LHS as

$$\binom{n-1}{k-1} \cdot \binom{n}{k+1} \cdot \binom{n+1}{k} = \frac{(n-1)!}{(k-1)!(n-1-k+1)!} \cdot \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{(n+1)!}{k!(n+1-k)!}$$

and the RHS as

$$\binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1} = \frac{(n-1)!}{k!(n-1-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{(n+1)!}{(k+1)!(n+1-k-1)!}$$

proving the equality holds. □

8) Prove

(i)  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ , and

(ii)

$$(x+y)^{\bar{n}} = \sum_{k=0}^{\bar{n}} \binom{\bar{n}}{k} x^{\bar{k}} y^{\bar{n}-k}$$

We can prove these by induction on  $n$ .

For (i):

Basecase ( $n=1$ )

$$\sum_{k=0}^1 \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^{1-0} + \binom{1}{1} x^1 y^0 = (x+y)^1$$

Inductive step

Suppose the equality holds for  $t$ ; WTS it holds for  $t+1$ . That is,

$$(x+y)^{t+1} = \sum_{k=0}^{t+1} \binom{t+1}{k} x^k y^{(t+1)-k}$$

## §2.2

8 (cont.ed)

We can write:

$$\begin{aligned}
\sum_{k=0}^{t+1} \binom{t+1}{k} x^k y^{t+1-k} &= \sum_{k=0}^{t+1} \left( \binom{t}{k} + \binom{t}{k-1} \right) x^k y^{t+1-k} \\
&= \sum_{k=0}^t \binom{t}{k} x^k y^{t+1-k} + \sum_{k=0}^t \binom{t}{k} x^{k+1} y^{t-k} \\
&= \sum_{k=0}^t \binom{t}{k} \left( x^k y^{t+1-k} + x^{k+1} y^{t-k} \right) \\
&= \sum_{k=0}^t \binom{t}{k} x^k y^{t-k} \left( (y-t+k) + (x-k) \right) \\
&= \sum_{k=0}^t \binom{t}{k} x^k y^{t-k} (y-t+x)
\end{aligned}$$

$$= (x+y-t)(x+y)^t$$

since  $\sum_{k=0}^t \binom{t}{k} x^k y^{t-k} = (x+y)^t$  by

inductive hypothesis

$$= (x+y)^{t+1}$$

(ii) is similar (proof by induction).

□