Math 455: Discrete Structures
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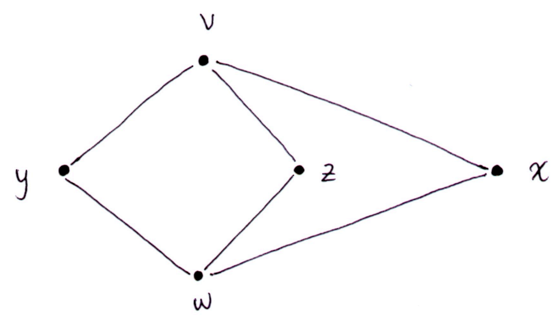
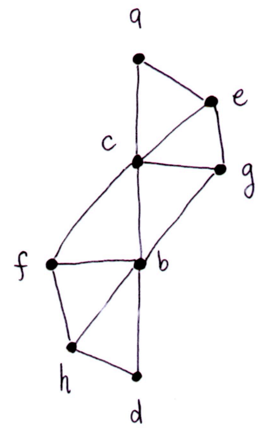
TA: Jennifer Li

Homework 4 Solutions

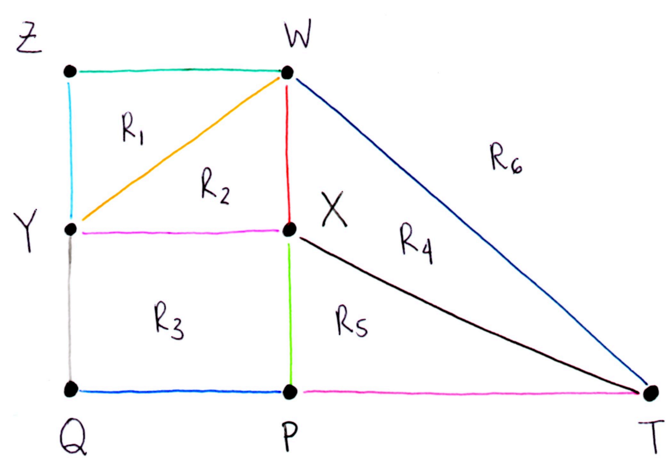
§1.5.1

1)





2) Let's label the original graph in the following way:



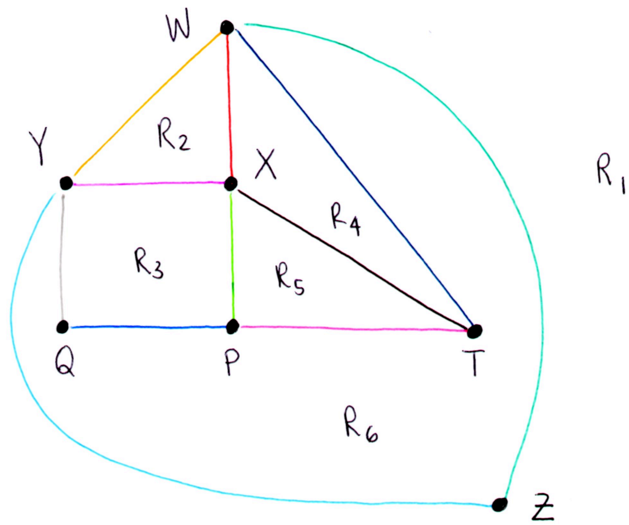
R_i = region # i, for $i = 1, 2, \ldots, 6$
Regions are bounded by edges
Ex R_2 is bounded by the yellow, red, and purple edges

§1.5.1

2) (cont.ed)

a) Make R_1 the exterior region

"Stretch" vertex Z so the graph looks like:



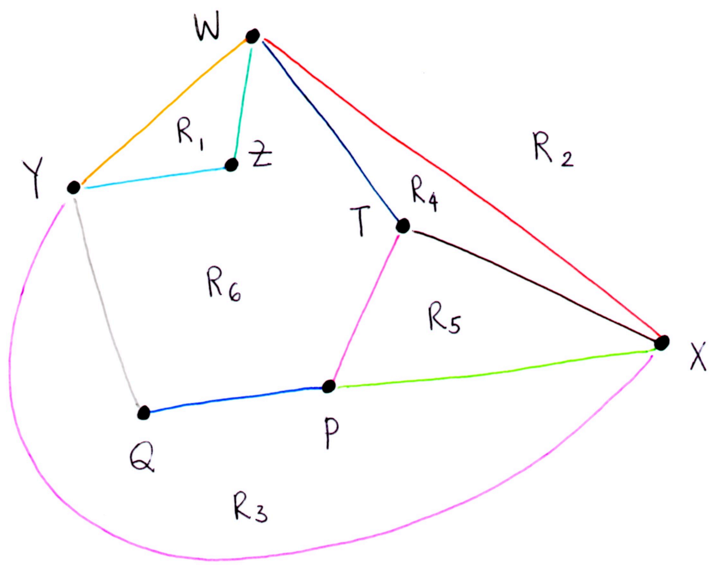
b) Make R_2 the exterior region

(i) "Fold" vertex Z inside ΔWXY

(ii) "Fold" vertex T inside ΔWXY

Notice that (ii) forces vertices P and Q to fold inside ΔWXY (so vertex X acts as a pivot).

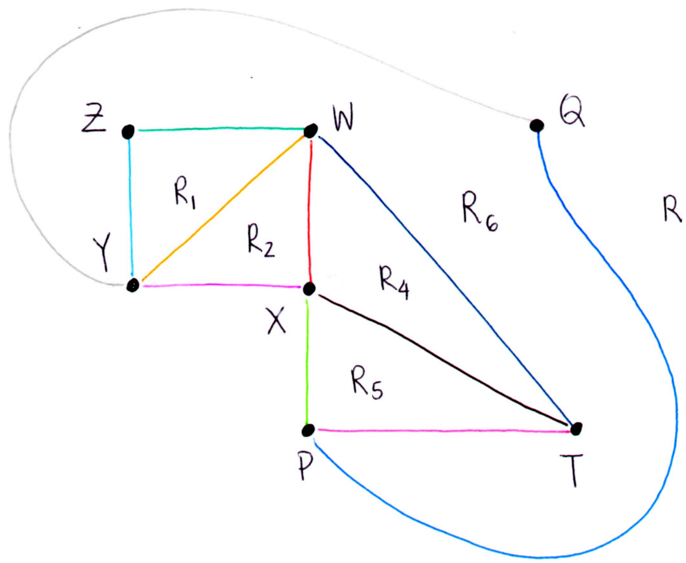
Then the graph looks like:



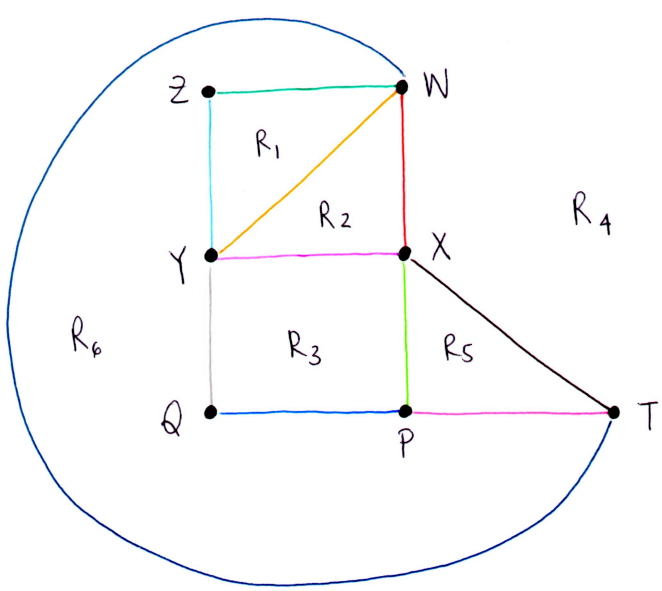
§1.5.1

2) (cont.ed)

c) Make R_3 the exterior region
"stretch" vertex Q to obtain the graph :



d) Make R_4 the exterior region.
"pull" the dark blue edge connecting vertices W and T over the rest of the graph to obtain :

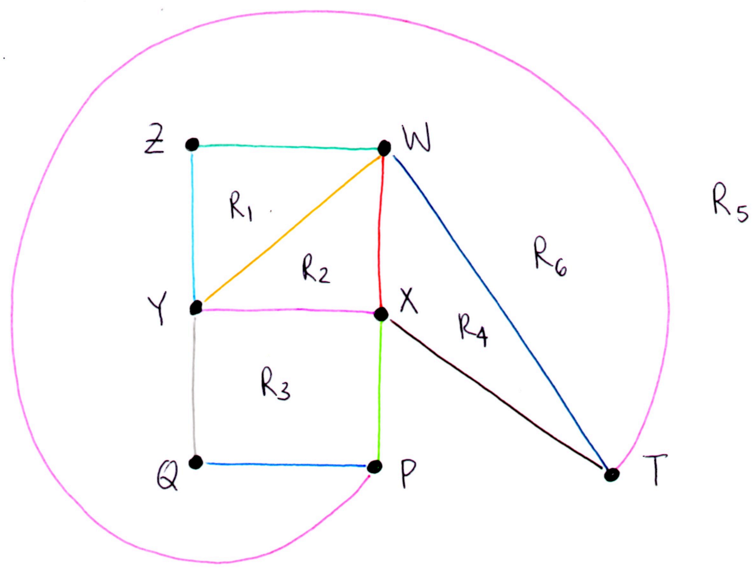


§1.5.1

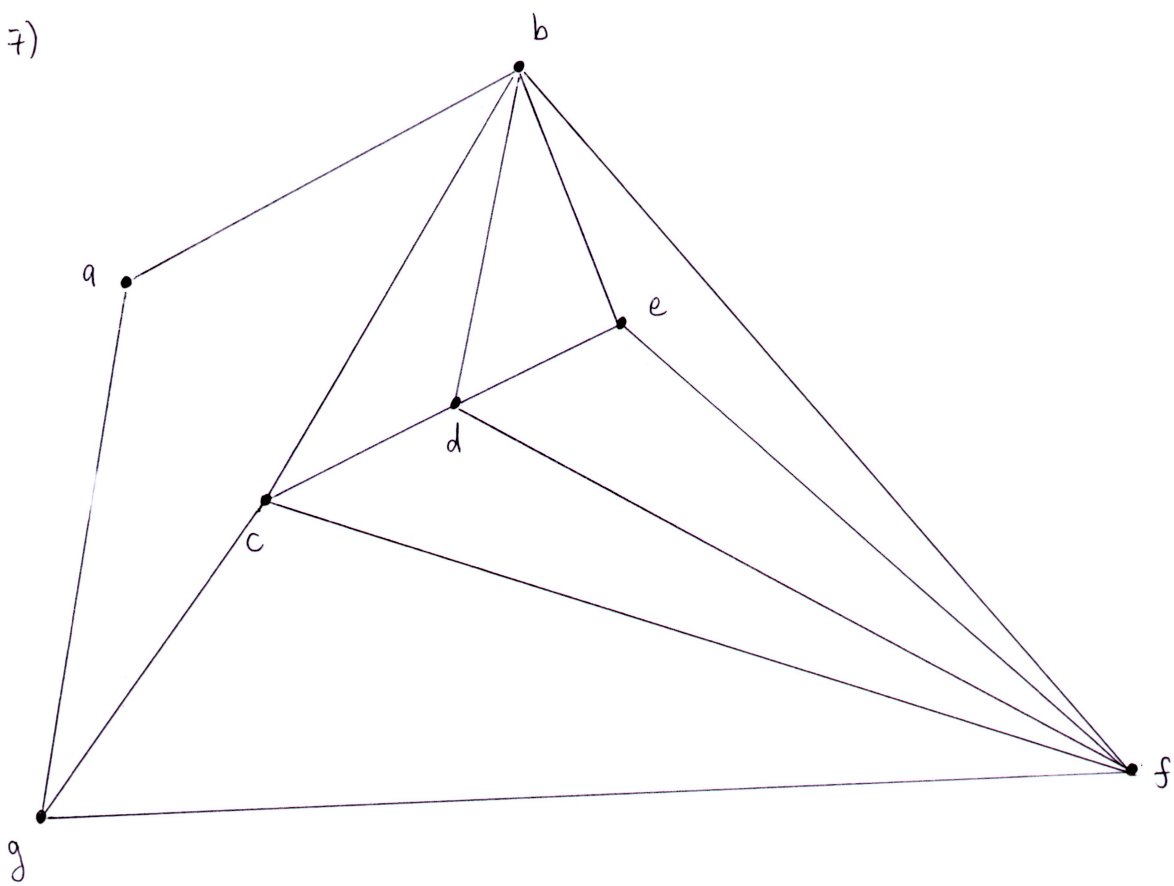
2) (cont.ed)

e) Make R_5 the exterior region

"pull" the pink edge connecting vertices P and T over the rest of the graph:



7)



§1.6.2

1) # of vertices in G = 24

Each vertex is of degree 3

$$\sum_{\text{vertex } v \in G} \deg(v) = 2 \cdot (\text{\# of edges in } G)$$

So $3(24) = 2 \cdot (\text{\# of edges in } G)$

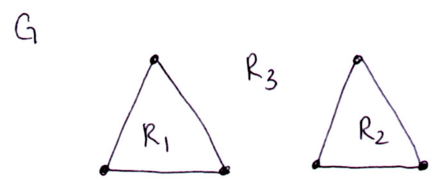
$\Rightarrow$ G has 36 edges

By Euler's Formula (Thm 1.31, p.78),

# of regions of $G = 2 - 24 + 36 = 14$

$\square$

3) Let G be the graph below:



Then $n = 6$

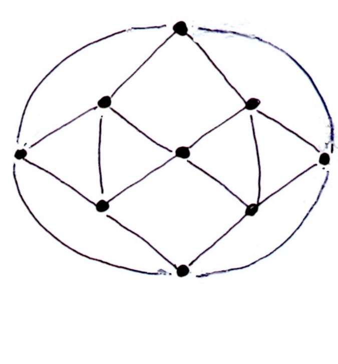
$q = 6$

$r = 3$

But $n - q + r = 6 - 6 + 3 = 3 \neq 2$.

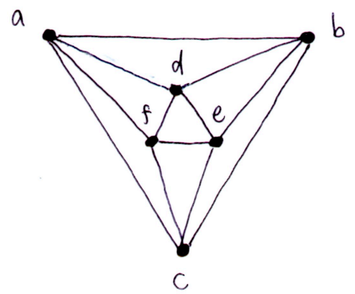
$\square$

10)

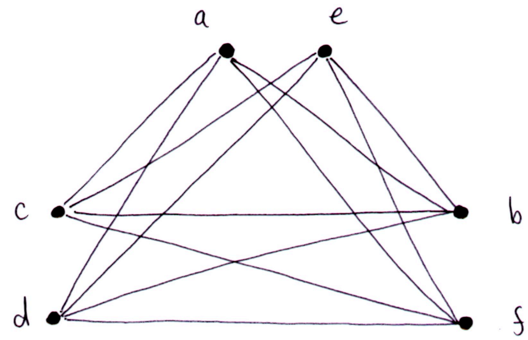


<u>§1.5.3</u>

1) Labelling the vertices of the octahedron as:

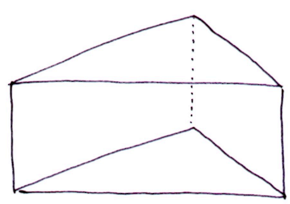


we may redraw the graph to see that it's a complete multipartite graph:

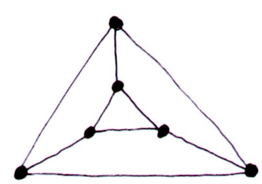


□

2) Prism



Graph



<u>§1.5.4</u>

2) Kuratowski's Theorem tells us:

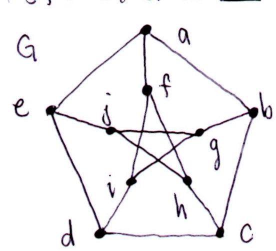
(i) If a graph G is planar, then G contains no subdivision of $K_{3,3}$ or K_5.

(ii) If a graph G contains no subdivision of $K_{3,3}$ or K_5, then G is planar.

The contrapositive of (i) is also true, that is:

If G <u>does</u> contain a subdivision of $K_{3,3}$ or K_5, then G is <u>not</u> planar.

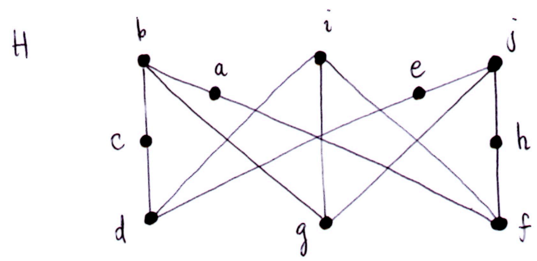
Suppose we label the vertices in G as:



§1.5.4

2) (cont.ed)

Now consider the graph H:



Then H is a subdivision of $K_{3,3}$ in G, so by Kuratowski's Theorem, G is not planar.

□

4) We first note that if $n \geq 5$, then G will contain a subdivision of K_5; in this case, Kuratowski's Theorem tells us that G is not planar.

Thus $n \leq 4$, so G must be of the following forms:

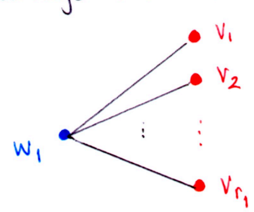
K_{r_1,r_2} or K_{r_1,r_2,r_3} or K_{r_1,r_2,r_3,r_4}

We also note that there can be at most one r_i such that $r_i \geq 3$ (otherwise G would contain a subdivision of $K_{3,3}$ and Kuratowski's Theorem would imply that G is not planar). So we have the following cases:

Case 1 $G = K_{r_1,r_2}$

Case 1a $G = K_{r_1,1}$ is planar for any $r_1 \in \mathbb{N}$

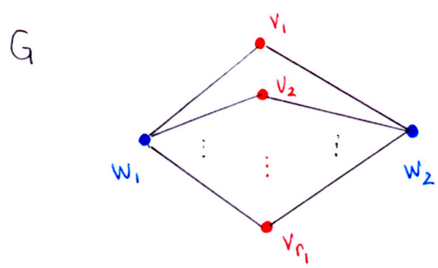
We can always draw G as:



$V = \{v_1, v_2, \dots, v_r\}$

$W = \{w_1\}$

Case 1b $G = K_{r_1,2}$ is planar for any $r_1 \in \mathbb{N}$



$V = \{v_1, v_2, \dots, v_r\}$

$W = \{w_1, w_2\}$

§1.5.4

4) (cont.ed)

Case 1c $G = K_{r_1, r_2}$ for $r_2 \geq 3$ is planar iff $r_1 \leq 2$.

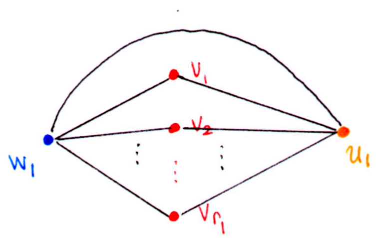
$G = K_{1, r_2} = K_{r_2, 1}$ and $G = K_{2, r_2} = K_{r_2, 2}$ are planar by Cases 1a and 1b, respectively.

But if $r_1 \geq 3$, then $G = K_{r_1, 3}$ will have a subdivision of $K_{3,3}$; by Kuratowski's Theorem, G will not be planar in this case.

Case 2 $G = K_{r_1, r_2, r_3}$

Case 2a $G = K_{r_1, 1, 1}$ is planar for any $r_1 \in \mathbb{N}$

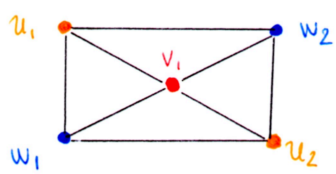
G



$V = \{v_1, v_2, \ldots, v_{r_1}\}$
$U = \{u_1\}$
$W = \{w_1\}$

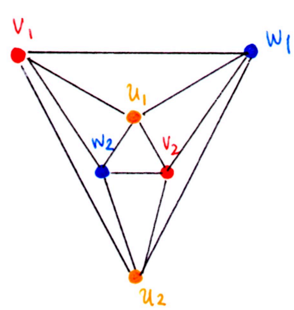
Case 2b $G = K_{r_1, 2, 2}$ is planar iff $r_1 = 1$ or $r_1 = 2$

$G = K_{1,2,2}$



$V = \{v_1\}$
$U = \{u_1, u_2\}$
$W = \{w_1, w_2\}$

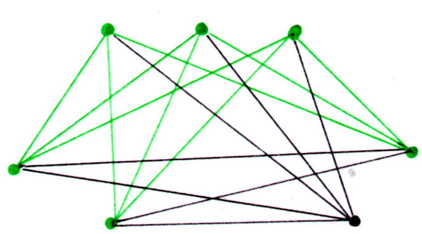
$G = K_{2,2,2}$ (see §1.5.3, exercise # 1)



$V = \{v_1, v_2\}$
$U = \{u_1, u_2\}$
$W = \{w_1, w_2\}$

But if $r_1 \geq 3$, we will have a subdivision of $K_{3,3}$, so G would not be planar:

$G = K_{3,2,2}$



$K_{3,3}$ in green

§1.5.4

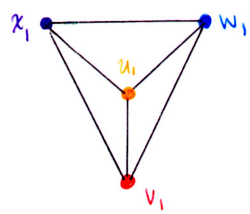
4) (cont.ed)

Case 2c $G = K_{r_1, r_2, r_3}$ where $r_2, r_3 \geq 3$ is not planar for any $r_1 \in \mathbb{N}$.
If $r_2, r_3 \geq 3$, then G contains $K_{3,3}$.

Case 3 $G = K_{r_1, r_2, r_3, r_4}$

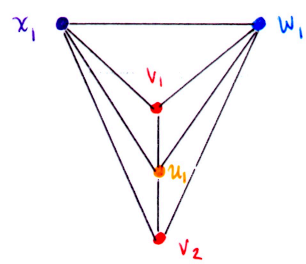
Case 3a $G = K_{r_1, 1, 1, 1}$ is planar iff $r_1 = 1$ or $r_1 = 2$

$G = K_{1,1,1,1}$



$V = \{v_1\}$
$U = \{u_1\}$
$W = \{w_1\}$
$X = \{x_1\}$

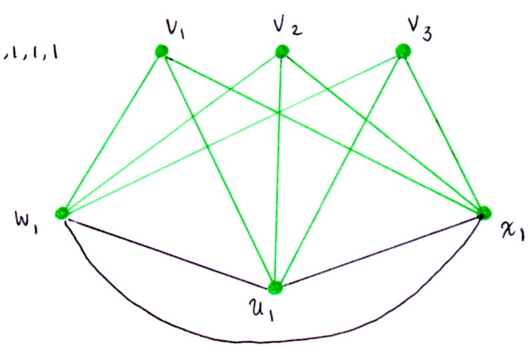
$G = K_{2,1,1,1}$



$V = \{v_1, v_2\}$
$U = \{u_1\}$
$W = \{w_1\}$
$X = \{x_1\}$

But if $r_1 \geq 3$, then G will contain $K_{3,3}$:

$G = K_{3,1,1,1}$



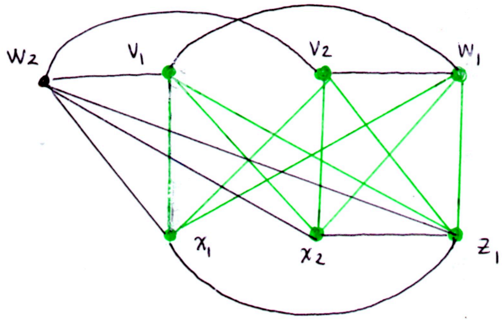
$V = \{v_1, v_2, v_3\}$
$U = \{u_1\}$
$W = \{w_1\}$
$X = \{x_1\}$

§1.5.4

4) (cont.ed)

Case 3b $G = K_{r_1,2,2,2}$ is not planar for any $r_1 \in \mathbb{N}$, since these contain $K_{3,3}$.

$G = K_{1,2,2,2}$



$V = \{v_1, v_2\}$
$W = \{w_1, w_2\}$
$X = \{x_1, x_2\}$
$Z = \{z_1\}$

Case 3c $G = K_{r_1,r_2,r_3,r_4}$ where $r_2, r_3, r_4 \geq 3$ is not not planar for any $r_1 \in \mathbb{N}$, since they contain $K_{3,3}$.

□

§1.6.1

1 a) G : tree

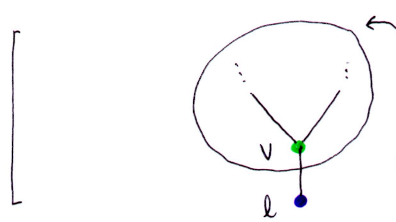
If $|V(G)| = 1$, then $\chi(G) = 1$.

Otherwise $\chi(G) = 2$; this can be proved by induction on $|V(G)|$:

Basis step If $|V(G)| = 2$, then we can 2-color G

inductive step Suppose that we can 2-color a tree with n-1 vertices. Let G be a tree with n vertices. Choose any leaf l of G, and color it purple. Suppose that l is adjacent to a vertex v. Then color v green.

tree G



If we remove l from G, then we get a tree of n-1 vertices, which can be 2-colored by our inductive hypothesis.

Then we just add l back to this (n-1)-tree to get a 2-colored tree of n vertices. By induction, any tree can be 2-colored.

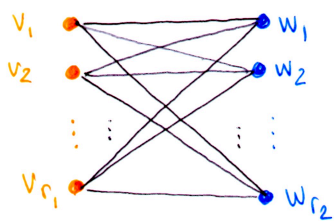
□

§1.6.1

1) (Cont.ed)

b) G : bipartite graph

Suppose $G = K_{r_1, r_2}$. Then $\chi(G) = 2$; we can just color all vertices of each set the same color, since vertices of the same set are not incident to one another.



$\square$

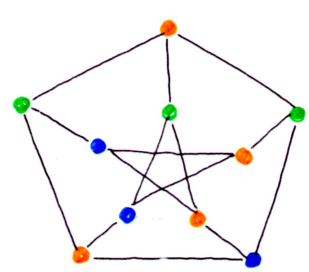
c) G : multipartite graph

Suppose $G = K_{r_1, r_2, \ldots, r_t}$. Then $\chi(G) = t$; because there are t sets and each vertex of a set is incident to every vertex of every other set, we need a minimum of t colors.

$\square$

d) G : Petersen graph

Since the Petersen graph contains an odd cycle, it is not 2-colorable. But we can color the vertices with 3 colors:



So $\chi(G) = 3$.

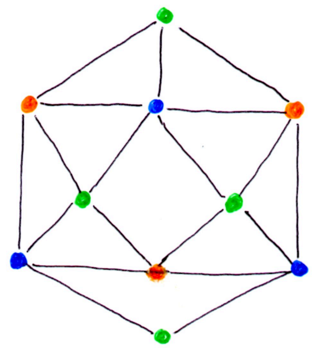
$\square$

§ 1.6.1

1) (cont.ed)

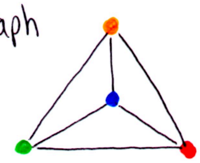
e) G : Birkhoff Diamond

$\chi(G) = 3$: (it is not 2-colorable since it contains an odd cycle)



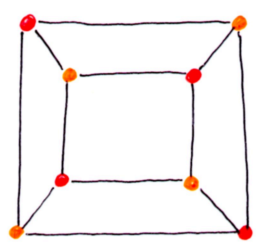
f) G : tetrahedron graph

$\chi(G) = 4$



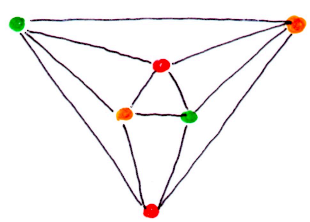
G: cube graph

$\chi(G) = 2$



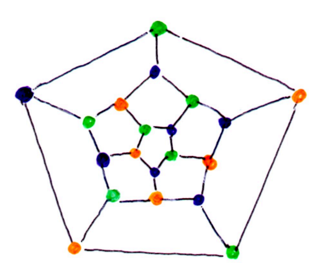
G: octahedron graph

$\chi(G) = 3$ (cannot be 2 because of odd cycles)



G: dodecahedron graph

$\chi(G) = 3$



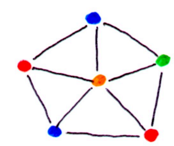
§1.6.1

1) (cont.ed)

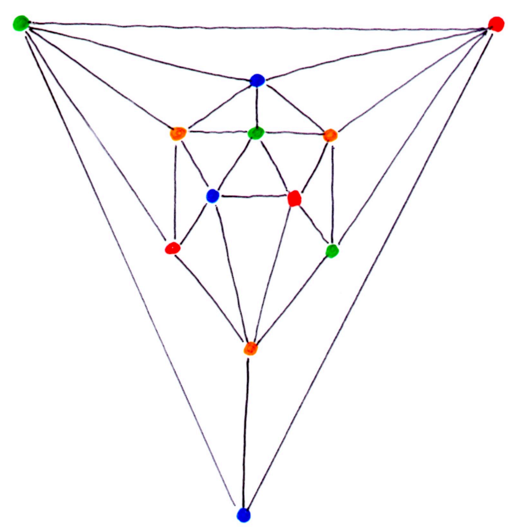
f) G: icosahedron graph

$\chi(G) \neq 2$ since there are odd cycles

$\chi(G) \neq 3$ because of this part in the graph:



But a 4-coloring works!

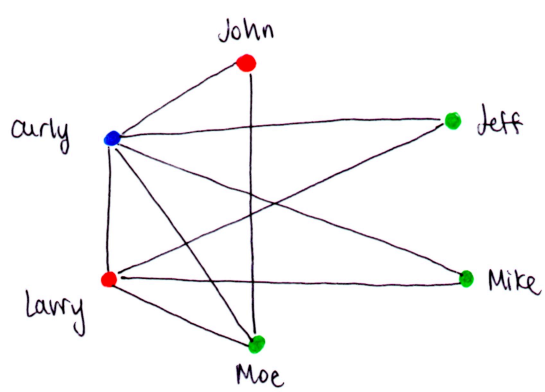


So $\chi(G) = 4$.

$\square$

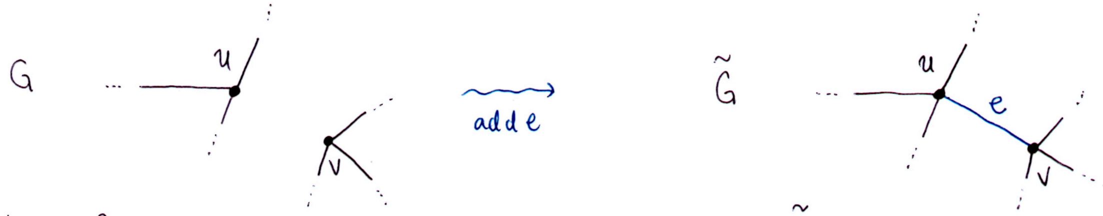
3) Three areas are required:

(edge between 2 vertices means that those vertices interfere and should be in separate regions):



§1.6.1

4) Suppose we start with a graph G and add an edge e between two vertices u and v to create a new graph $\tilde{G}$. The pictures below show an example with sample colors.

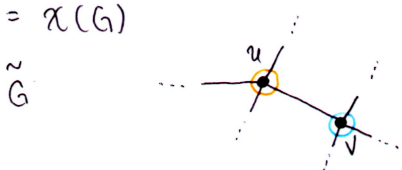


Colors in G: ● ● ● ● ●

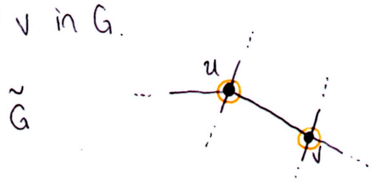
Colors in $\tilde{G}$: ● ● ● ● ● + ?

Case 1 If the colors of u and v were different in G, then the addition of e does not introduce any conflicts, so nothing changes.

$\Rightarrow \chi(\tilde{G}) = \chi(G)$



Case 2 Suppose that the colors of u and v were the same in G. Then addition of e causes a conflict. We will need to change either the color of u or the color of v in $\tilde{G}$.

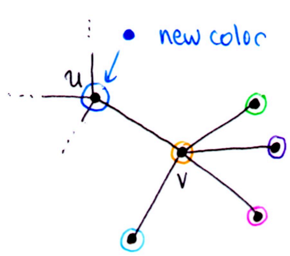


WLOG let's change the color of u.

Let's look at vertex v.

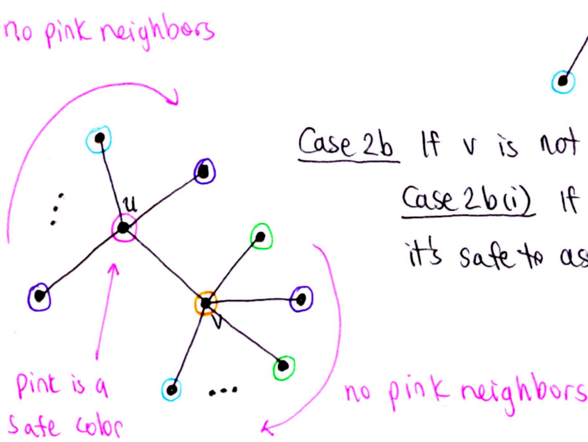
Case 2a If v is adjacent to a vertex of every other color in G, then we are forced to assign u to a new color, increasing the chromatic number by 1:

$$\chi(\tilde{G}) = \chi(G) + 1$$



Case 2b If v is not adjacent to a vertex of one color in G, then we check the neighbors of u.

Case 2b(i) If u is not adjacent to a vertex of this same color, then it's safe to assign this color to u in $\tilde{G}$. Then $\chi(\tilde{G}) = \chi(G)$.

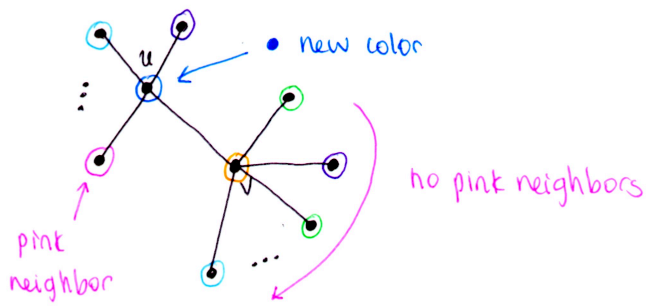


§1.6.1

4) (cont.ed)

Case 2b(ii) If u is adjacent to a vertex of this same color already, then we are unable to assign u to this color.

If we test all colors that satisfy case 2b and end up in case 2b(ii) for each one of these, then we are forced to assign u a new color.



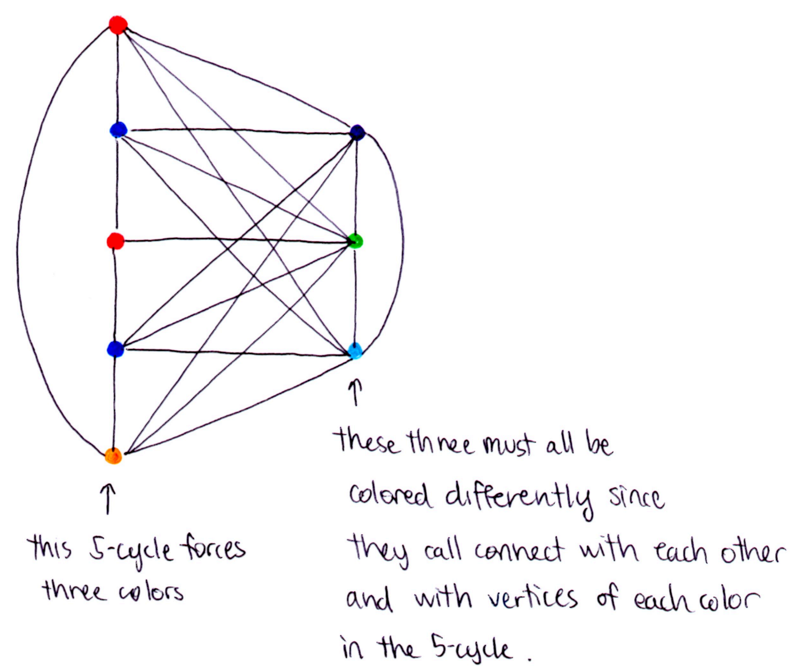
Then $\chi(\tilde{G}) = \chi(G) + 1$.

Since this covers all cases, we see that $\chi(\tilde{G}) \leq \chi(G) + 1$.

$\square$

§1.6.2

2) $\chi(G) = 6$:



§1.6.2

2) (Cont.ed)

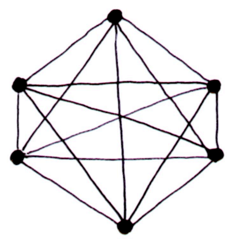
$\omega(G) = 5$:

By Theorem 1.44 (p.92), we know that $\chi(G) \geq \omega(G)$.

So $\omega(G) \leq 6$.

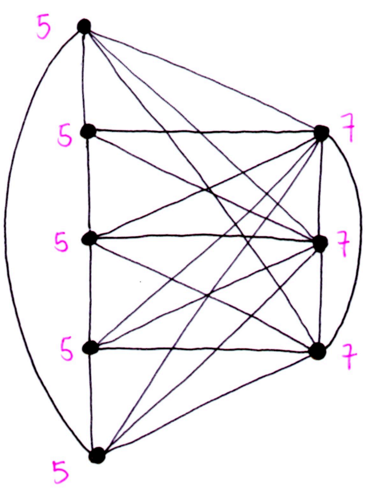
In order to have K_6 as a subgraph, we need to pick out six vertices in G that are all adjacent to each other and each must be of degree ≥ 5 in G:

K_6



Consider the degrees of vertices in G:

G



Suppose G has a K_6 subgraph (we'll show this is false). If we pick all three of the degree 7 vertices, then we must choose two degree 5 vertices to be in our K_6 subgraph. But the degree 5 vertices are all linked together, and since they are of <u>exactly</u> degree 5, we need to include all of them in our subgraph. Now our K_6 subgraph has $\underset{\text{deg } 7}{\underset{\uparrow}{3}} + \underset{\text{deg } 5}{\underset{\uparrow}{5}} = 8$ vertices! $\rightarrow\leftarrow$

cannot happen!

If we only choose one or two of the degree 7 vertices, then we must include some degree 5 vertices, which are all linked so all degree 5 vertices must be included. But degree 5 vertices will drag in all degree 7 vertices, so again our K_6 subgraph has 8 vertices $\rightarrow\leftarrow$

If we only choose degree 5 vertices — again they will force inclusion of degree 7 vertices in our K_6, making an 8-vertex K_6 $\rightarrow\leftarrow$

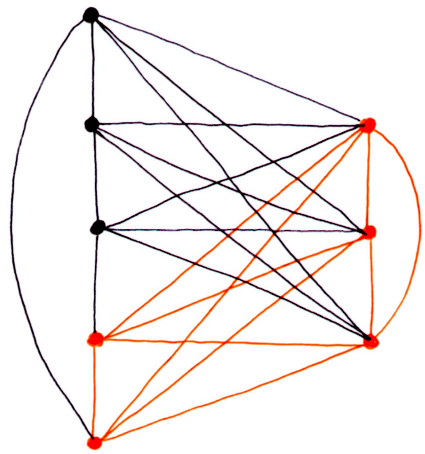
Thus $\omega(G) \neq 6$.

However we can find K_5 in G:

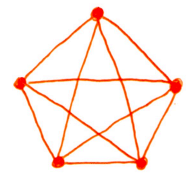
§1.6.2

2) (cont.ed)

G



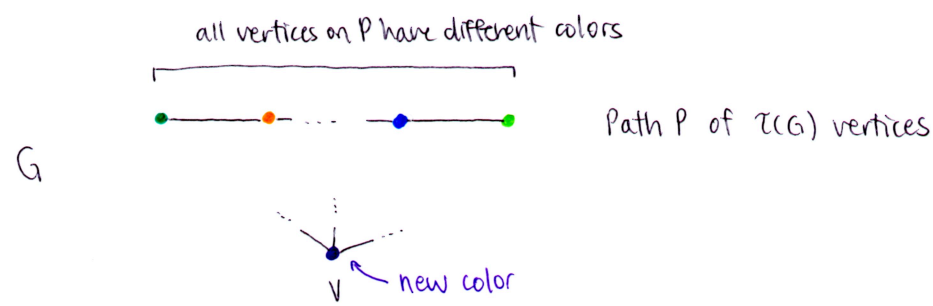
K_5 subgraph



$\square$

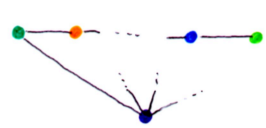
4) Suppose by contradiction that $\underbrace{\tau(G)}_{\substack{\shortparallel \\ \text{length of longest path in } G}} < \chi(G)$.

Say P is a path of length $\tau(G)$. Since we assumed that $\tau(G) < \chi(G)$, not every color used to color all of G is included in the coloring of P. In other words, there exists a vertex v in G that does not lie on path P and is of a different color than all vertices on P.



But since v is of a different color than all vertices on P, vertex v must be adjacent to every vertex on P. Reason: if this were not true, i.e. if v were <u>not</u> adjacent to some vertex p_i on P, then we could have just colored vertices p_i and v the same color, using less colors than $\chi(G)$ $\rightarrow\leftarrow$

Then we have a path of length one greater than length of P. But we assumed that P is a path of maximal length in G $\rightarrow\leftarrow$

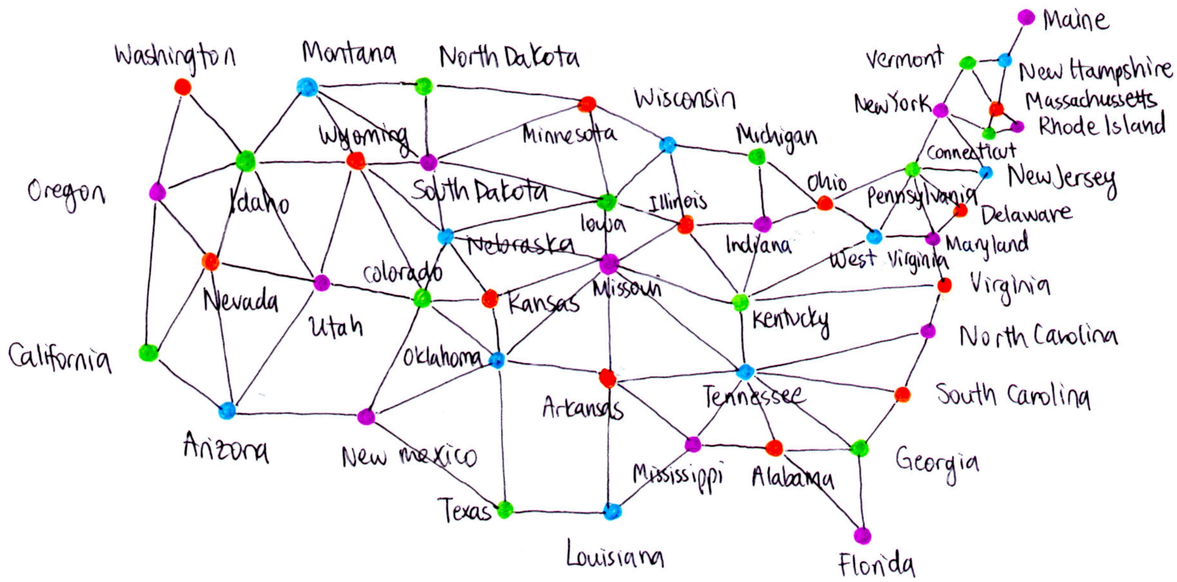


$\therefore \chi(G) \leq \tau(G)$.

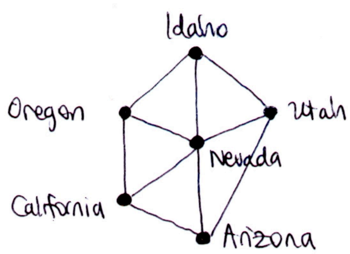
$\square$

§1.6.3

1) $\chi(G) = 4$ (where G = map of the United States)

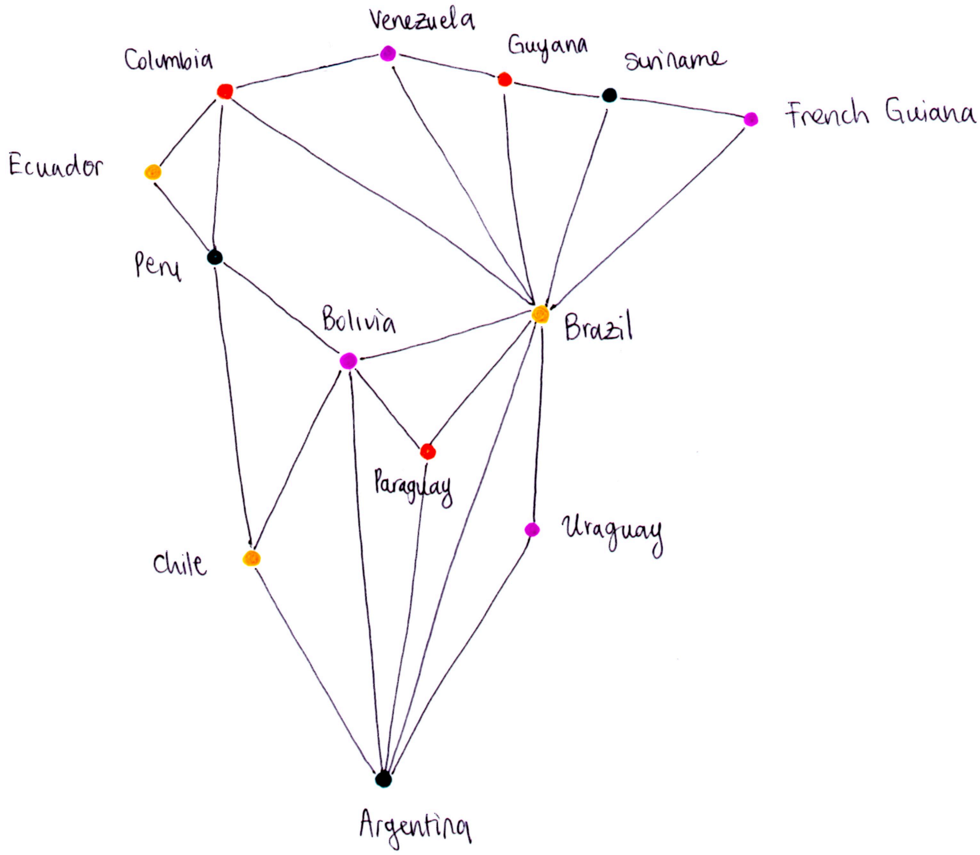


We need $\chi(G) \geq 4$ because this portion of the map requires 4 colors:

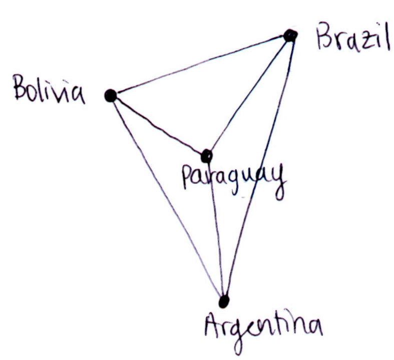


§1.6.3

2) $\chi(G) = 4$ (where G = map of South America)

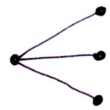


We need $\chi(G) \geq 4$ because this portion of the map requires 4 colors:



§1.6.4

1 a) $K_{1,3}$

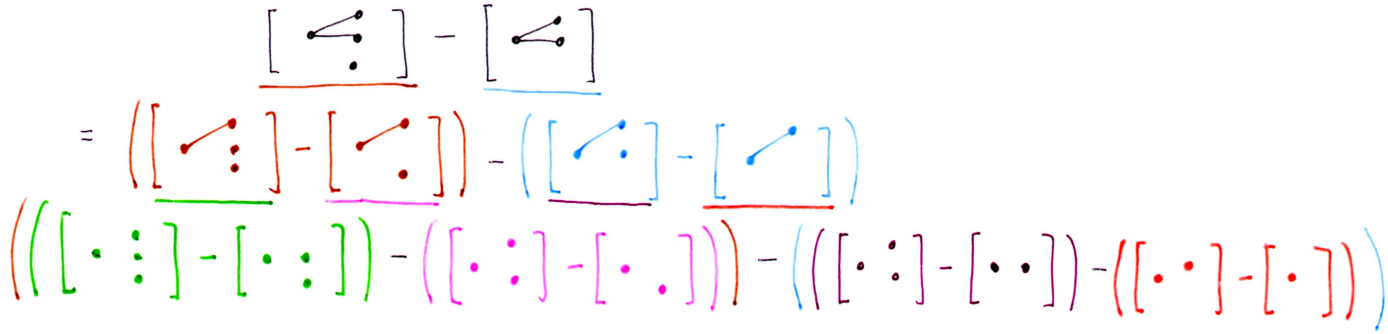


Iteration # — Graph representation of $C_G(k)$

1

2

3



Then $C_G(k) = \left((k^4 - k^3) - (k^3 - k^2)\right) - \left((k^3 - k^2) - (k^2 - k)\right)$

$= (k^4 - 2k^3 + k^2) - (k^3 - 2k^2 + k)$

$= k^4 - 3k^3 + 3k^2 - k$

The number of 5-colorings of G equals $C_G(5)$

$\qquad\qquad \parallel$

$5^4 - 3(5)^3 + 3(5)^2 - 5 = 180$

Basic Idea:

- Start with given graph G
- In each iteration (or step), we look at the "piece(s)" from the previous step and do the following for each piece:
  ↳ I drew these in brackets
  - Pick any edge e from that piece G
  - Create two new pieces from G:
    1) $G - e$
       To get this, we just remove edge e from G
    2) G/e
       To get this, we "collapse the endpoints of e to one vertex, and remove any multiple edges that might show up.

§1.6.4

1) (cont.ed) Parts (b) - (f) are done similarly to (a)

(b) $k^6 - 5k^5 + 10k^4 - 10k^3 + 5k^2 - k$

(c) $k^4 - 4k^3 + 6k^2 - 3k$

(d) $k^5 - 5k^4 + 10k^3 - 10k^2 + 4k$

(e) $k^4 - 5k^3 + 8k^2 - 4k$

(f) $k^5 - 9k^4 + 29k^3 - 39k^2 + 18k$

$\square$

2) if we plug in $k=2$, we get

$$2^4 - 4(2)^3 + 3(2)^2 = -4 < 0$$

But chromatic polynomials at positive integers should always give nonnegative values. So this cannot be a chromatic polynomial for any graph.

$\square$