Math 455: Discrete structures
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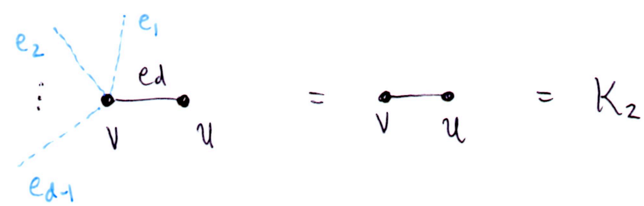
Homework 3 solutions

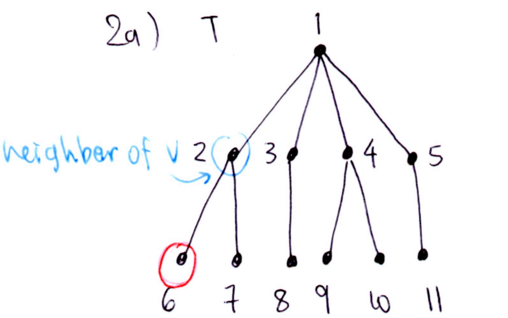
§1.3.4

1) The leaves are removed in Step 4 of the algorithm.
Let v be a vertex. Suppose deg(v) = d.
If v is one of the last two remaining vertices, then d-1 of its neighbors were removed. Then it must appear in the sequence d-1 times.



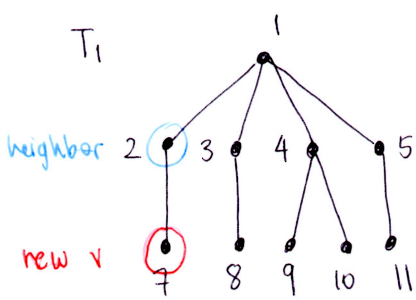
----- removed edge

2a) T



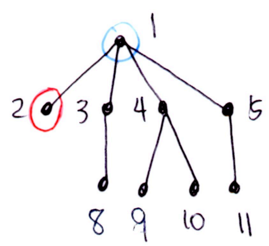
$i = 0$
$T_0 = T$
Choose vertex v (leaf w/ smallest label)
Sequence: 2
neighbor's label

T_1



$i = 1$
$T_1 = T_0 - v$
Choose new v
Sequence: 2, 2

T_2



$i = 2$
$T_2 = T_1 - v$
Choose new v
Seq: 2, 2, 1

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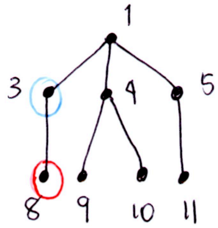
2a) (cont.ed)

T_3

new v



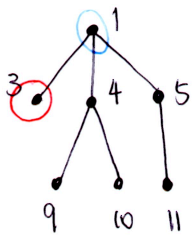
$i=3$

$T_3 = T_2 - v$

Choose new v

Seq: 2, 2, 1, 3

T_4

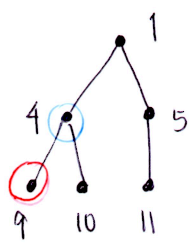


$i = 4$

$T_4 = T_3 - v$

Seq: 2, 2, 1, 3, 1

T_5

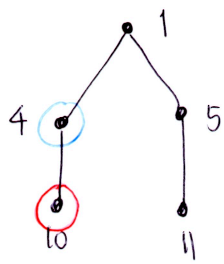


$i=5$

$T_5 = T_4 - v$

Seq.: 2, 2, 1, 3, 1, 4

T_6

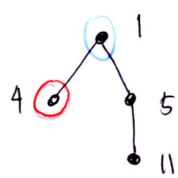


$i=6$

$T_6 = T_5 - v$

Seq.: 2, 2, 1, 3, 1, 4, 4

T_7

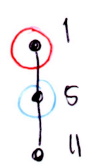


$i=7$

$T_7 = T_6 - v$

Seq: 2, 2, 1, 3, 1, 4, 4, 1

T_8



$i=8$

$T_8 = T_7 - v$

Seq: 2, 2, 1, 3, 1, 4, 4, 1, 5

T_9

$i=9$

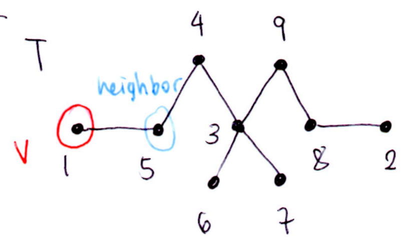
$T_9 = T_7 - v = K_2$ stop here!

∴ the Prüfer sequence is: 2, 2, 1, 3, 1, 4, 4, 1, 5

□

§1.3.4

2 b) T



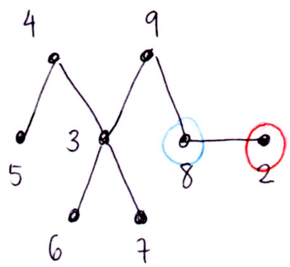
$i=0$

$T_0 = T$

New v

Seq: 5

T_1

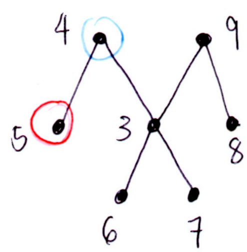


$i=1$

$T_1 = T_0 - v$

Seq: 5, 8

T_2

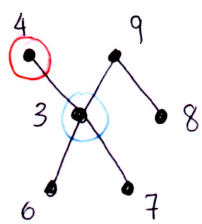


$i=2$

$T_2 = T_1 - v$

Seq: 5, 8, 4

T_3

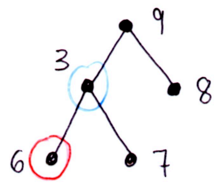


$i=3$

$T_3 = T_2 - v$

Seq: 5, 8, 4, 3

T_4

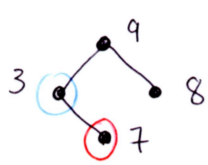


$i=4$

$T_4 = T_3 - v$

Seq: 5, 8, 4, 3, 3

T_5

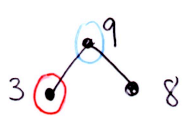


$i=5$

$T_5 = T_4 - v$

Seq: 5, 8, 4, 3, 3, 3

T_6



$i=6$

Seq: 5, 8, 4, 3, 3, 3, 9

T_7

$T_7 = K_2$ stop here!

∴ the Prüfer sequence is 5, 8, 4, 3, 3, 3, 9

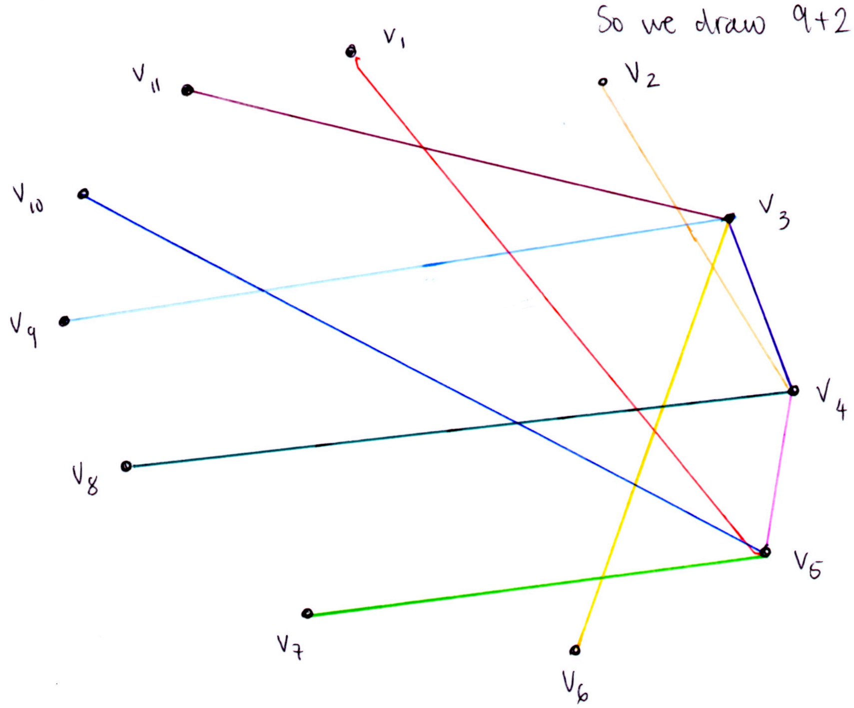
□

§1.3.4

3) Seq : $\overbrace{5, 4, 3, 5, 4, 3, 5, 4, 3}^{\sigma}$

total :

So we draw $9+2 = 11$ vertices



① Let $i=0$, and

$\sigma_0 = \sigma = 5, 4, 3, 5, 4, 3, 5, 4, 3$ — length of $\sigma = 9$

$S_0 = S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ — ((length of σ) + 2)

Let j = smallest # in S_0 that is not in seq. σ_0 $= 1$ } draw an edge between v_1 and v_5

First # in seq. σ_0 is 5

Create $\underbrace{\sigma_{0+1}}_{\sigma_1} = \sigma_0 - \underbrace{(\text{first \# in } \sigma_0)}_{5} = 4, 3, 5, 4, 3, 5, 4, 3$

Create $\underbrace{S_{0+1}}_{S_1} = S_0 - \underbrace{j}_{1} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

② Let $i = 1$

j = smallest # in S_1 that is not in σ_1 $= 2$ } draw edge between v_2 and v_4

First # in seq. σ_1 is 4

Create $\sigma_2 = \sigma_1 - (\text{first \# in } \sigma_1) = 3, 5, 4, 3, 5, 4, 3$

create $S_2 = S_1 - \underbrace{j}_{2} = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$

ps

§1.3.4

3) (cont.ed)

③ Let $i = 2$

$j = 6$
First # in $\sigma_2 = 3$ } edge between v_6 and v_3

Create $\sigma_3 = \sigma_2 - 3 = 5, 4, 3, 5, 4, 3$

Create $S_3 = S_2 - \underbrace{j}_{6} = \{3, 4, 5, 7, 8, 9, 10, 11\}$

④ $i = 3$

$j = 7$
First # in $\sigma_3 = 5$ } edge from v_7 to v_5

Create $\sigma_4 = \sigma_3 - 5 = 4, 3, 5, 4, 3$

Create $S_4 = S_3 - \underbrace{j}_{7} = \{3, 4, 5, 8, 9, 10, 11\}$

⑤ $i = 4$

$j = 8$
First # in $\sigma_4 = 4$ } edge from v_8 to v_4

Create $\sigma_5 = \sigma_4 - 4 = 3, 5, 4, 3$

Create $S_5 = S_5 - \underbrace{j}_{8} = \{3, 4, 5, 9, 10, 11\}$

⑥ $i = 5$

$j = 9$; First # in $\sigma_5 = 3$ $\Rightarrow$ edge from v_9 to v_3

Create $\sigma_6 = \sigma_5 - 3 = 5, 4, 3$

Create $S_6 = S_5 - \underbrace{j}_{9} = \{3, 4, 5, 10, 11\}$

⑦ $i = 6$

$j = 10$; First # in $\sigma_6 = 5$ $\Rightarrow$ edge from v_{10} to v_5

Create $\sigma_7 = \sigma_6 - 5 = 4, 3$

Create $S_7 = S_6 - \underbrace{j}_{10} = \{3, 4, 5, 11\}$

⑧ $i = 7$

$j = 5$; First # in $\sigma_7 = 4$ $\Rightarrow$ edge from v_4 to v_5

Create $\sigma_8 = \sigma_7 - 4 = 3$

Create $S_8 = S_7 - \underbrace{j}_{5} = \{3, 4, 11\}$

§1.3.4

3) (cont.ed)

⑨ $i = 8$

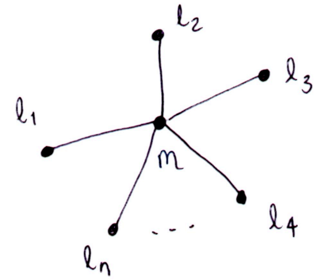
$j = 4$; first # in $\sigma_8 = 3$ $\Rightarrow$ connect v_4 and v_3

Create $\sigma_9 = \sigma_8 - 3 = \emptyset$

Since σ_9 is empty, we draw an edge between the remaining two vertices in $S_9 = S_8 - \underset{4}{j} = \{3, 11\}$ $\square$

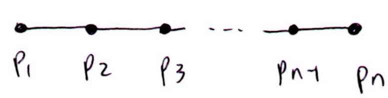
4) Star graphs have constant Prüfer sequences or $K_{1,n}$

These graphs look like:



Every vertex except the middle vertex m is a leaf. So each leaf vertex l_i will be chosen as "vertex v" from the algorithm, and each time, the neighboring vertex will be m. Then in the Prüfer sequence, only the weight / label of vertex m will be recorded. $\square$

5) Paths have distinct entries in the Prüfer sequence.



This is because each vertex will eventually become a leaf. Every vertex except vertices p_1 and p_n will be the neighbor of a leaf, so those weights (distinct numbers in $\{1, \ldots, n\}$) will appear in the Prüfer sequence. $\square$

§1.3.4

6) Let e be an edge of K_n. We want to show that the number of spanning trees of $K_n - e$ is $(n-2) \cdot n^{n-3}$.

K_n has a total of
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} \text{ edges,}$$
and any spanning tree of K_n has $n-1$ edges.

Then the proportion of edges in a spanning tree to edges in K_n is
$$\frac{n-1}{\left(n!/2!(n-2)!\right)} = n-1 \cdot \frac{2}{n(n-1)}$$
$$= \frac{2}{n}$$

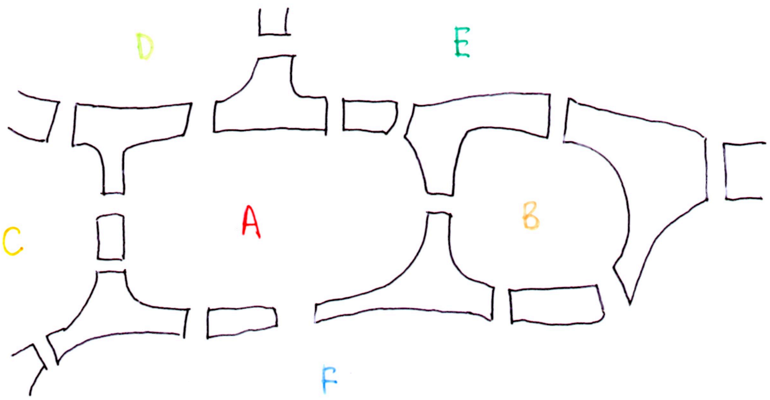
This means that edge e appears in $\frac{2}{n}$ of all spanning trees, and does not appear in $(n-2)/n$ of spanning trees.

By Cayley's Theorem (Thm 1.18, p 44), there are n^{n-2} spanning trees.

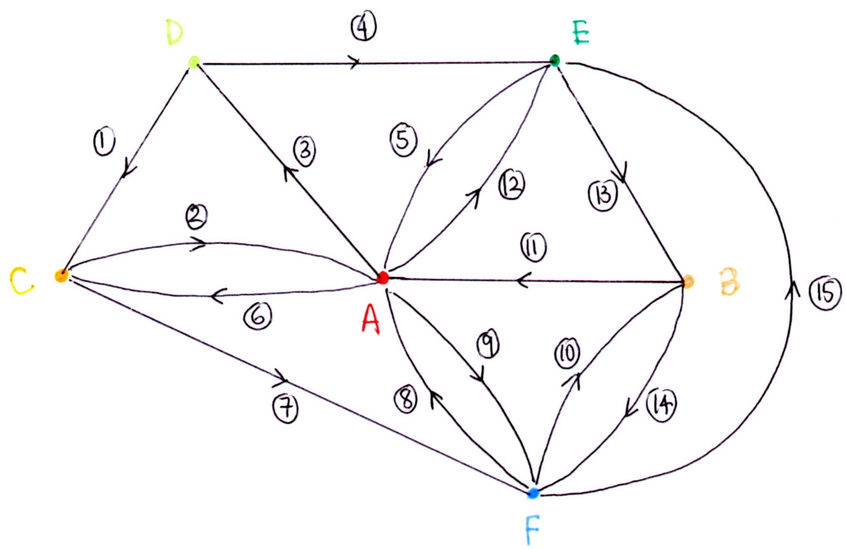
Then the number of spanning trees that do not contain e (i.e., # of spanning trees of $K_n - e$), equals
$$\frac{n-2}{n} \cdot n^{n-2} = (n-2) \cdot n^{n-3}$$
$\square$

§1.4.1

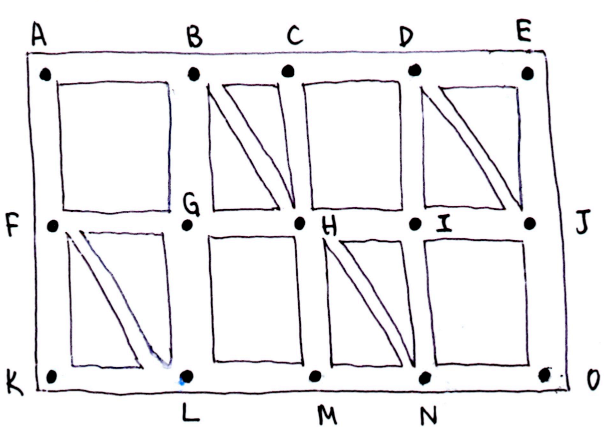
3) Yes; the route is numbered below:



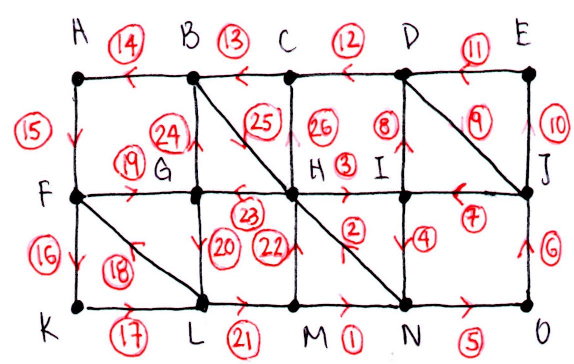
↓ drawn as a graph



4) Yes;

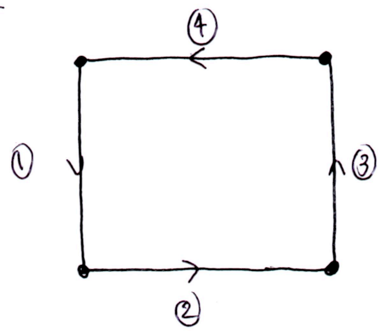


graph →



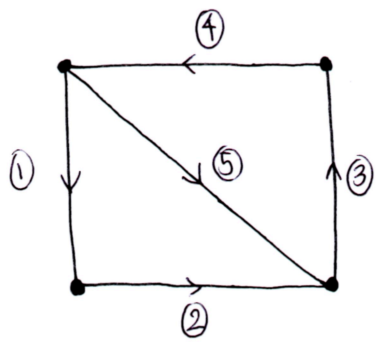
§1.4.2

1 a)



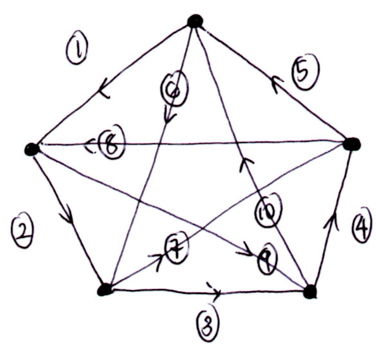
4 vertices
4 edges

b)



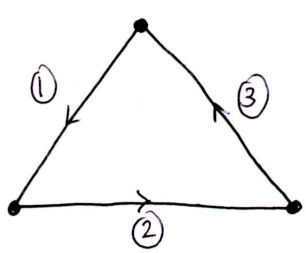
4 vertices
5 edges

c)



5 vertices
10 edges

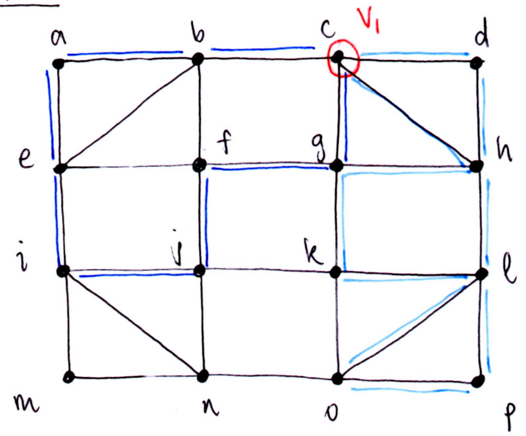
d)



3 vertices
3 edges

§1.4.2

2)



$R_1 = a, b, c, g, f, j, i, e, a$

$i = 1$

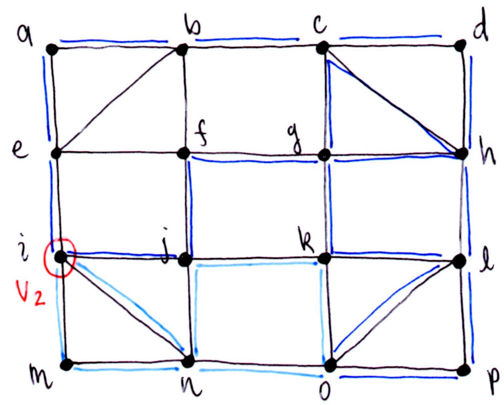
Choose $v_1 = c$

- It is on R_1 ✓
- It is incident with unmarked edge $e_1 = [\underset{\text{start vertex}}{c}, \underset{\text{end vertex}}{d}]$

Build circuit: $Q_1 = c, d, h, l, p, o, l, k, g, h, c$

Now let $R_2 = a, b, \underbrace{\overset{v_1}{c}, d, h, l, p, o, l, k, g, h, \overset{v_1}{c}}_{Q_1 \text{ inserted into } R_1 \text{ at vertex } c}, g, f, j, i, e, a$

Set $i = 1 + 1 = 2$



R_2: ——

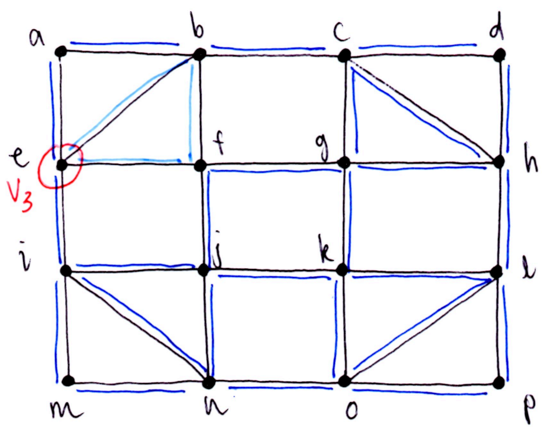
Choose $v_2 = i$ (incident to unmarked edge $e_2 = [i, m]$)

Build circuit:

$Q_2 = i, m, n, o, k, j, n, i$

Now let $R_3 = a, b, c, d, h, l, p, o, l, k, g, h, c, g, f, j, \underbrace{i, m, n, o, k, j, n, i}_{Q_2}, e, a$

Set $i = 2 + 1 = 3$



R_3: ——

Choose $v_3 = e$ (incident to unmarked edge $e_3 = [e, b]$)

Build circuit:

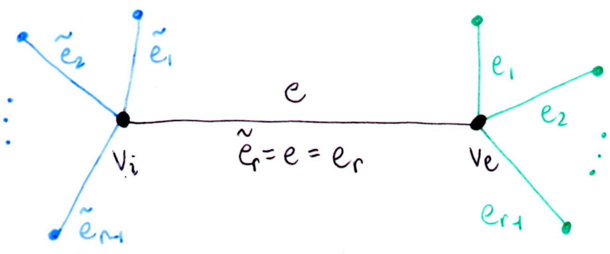
$Q_3 = e, b, f, e$

Let $R_4 = a, b, c, d, h, l, p, o, l, k, g, h, c, g, f, j, i, m, n, o, k, j, n, i, \overbrace{e, b, f, e}^{Q_3}, a$

Since all edges appear in R_4, we are done; R_4 is a Eulerian circuit.

§1.4.2

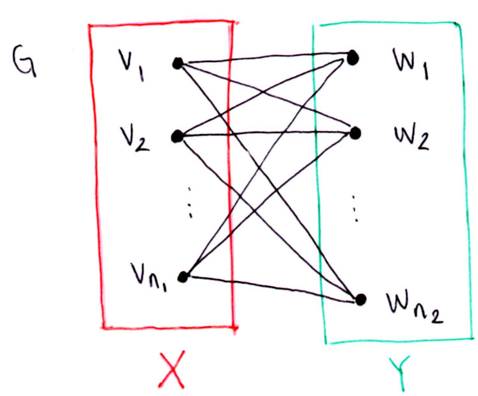
6) Let G be a connected graph which is regular of degree r. Then every edge e in G shares the same initial vertex with (r-1) other edges, and it also shares its end vertex with (r-1) other edges :



v_i = initial vertex of e
v_e = end vertex of edge e

$\deg(v_i) = r$
$\Rightarrow$ v_i is incident to r edges total : $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{r-1}, e = \tilde{e}_r$
$\deg(v_e) = r$
$\Rightarrow$ v_e is incident to r edges : $e_1, e_2, \ldots, e_{r-1}, e = e_r$
Then edge e has a total of $2(r-1)$ neighboring edges in graph G.
In L(G), edge e becomes a vertex with $2(r-1)$ adjacent vertices (definition of line graph). So every vertex in L(G) has even degree.
Then by Theorem 1.20 (p.56), this is equivalent to saying that G is Eulerian. (Note: since G is connected, so is L(G)). □

7) Let $G = K_{n_1, n_2}$.
a) Let's suppose the bipartite graph G has vertices that can be separated into sets X and Y, where $|X| = n_1$ and $|Y| = n_2$. So the general picture for G looks like:



§1.4.2

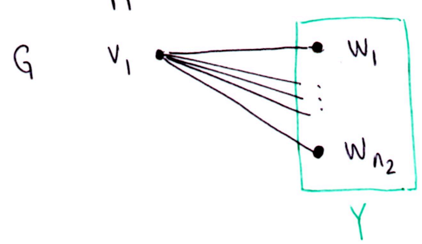
7 a) (Cont.ed)

The main corollary we use: (Cor 1.21, p.57) The connected graph G contains an Eulerian trail if and only if there are at most 2 vertices of odd degree.

We consider this in cases:

Case 1 Suppose that one of n_1 or n_2 is 1. WLOG let's suppose that $n_1 = 1$.

Case 1a Suppose that $n_2 = |Y|$ is even. So G looks like



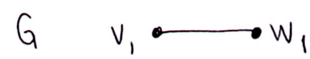
Since n_2 is even, we know that v_1 is adjacent to an even number of vertices. But each vertex in Y is only adjacent to v_1, i.e., $\underbrace{\deg(w_i) = 1}_{\text{odd}}$ for each $w_i \in Y$.

By Cor 1.21, in order for G to have an Eulerian trail, we cannot have more than two vertices in Y.

So in the case where $G = K_{1,\text{even}}$, the only way to have a Eulerian trail in G is if $n_2 = 2$.

Case 1b Suppose that $n_2 = |Y|$ is odd.

The picture for G looks the same as in Case 1a, except now vertex v_1 has odd degree.

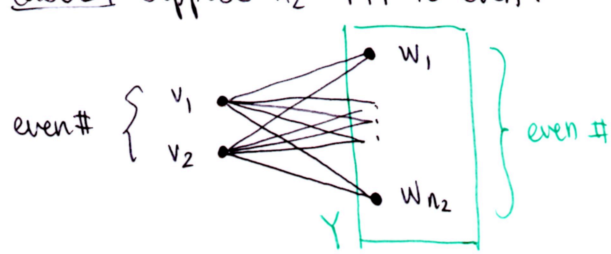
If $n_2 = 1$, then G has a Eulerian trail: 

If n_2 is any odd # greater than 3, we have more than two vertices of odd degree, so by Cor 1.21, G is not Eulerian.

Thus for the case where $G = K_{1,\text{odd}}$, the only way to get a Eulerian trail is if $n_1 = 1$.

Case 2 Suppose that one of n_1 or n_2 is 2. WLOG suppose $n_1 = 2$.

Case 2a Suppose $n_2 = |Y|$ is even.



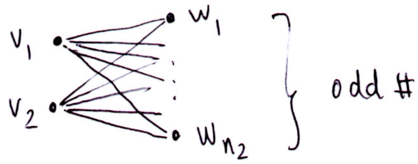
Then every vertex in G has even degree, so Cor 1.21 implies that G has an Eulerian trail.

So if $G = K_{2,n_2}$, where n_2 is even, then G will always have a Eul. trail.

§1.4.2

7a) (cont'ed)

Case 2b Suppose $n_2 = |Y|$ is odd.



Note that each vertex $w_i \in Y$ has degree 2, and both v_1 and v_2 have odd degree n_2. But only two vertices have odd degree so G has a Eulerian trail.

Thus $G = K_{2,\text{odd}}$ has a Eul. trail for <u>any</u> odd number n_2.

Case 3 Suppose that $n_1 \geq 3$ and $n_2 \geq 3$.

Case 3a If both n_1 and n_2 are even, then every vertex has even degree. So $G = K_{\text{even},\text{even}}$ will always have a Eulerian trail, for any n_1, n_2 even.

Case 3b If one of n_1 or n_2 is odd, then we have more than 3 vertices of odd degree, so G does not have Eul. trail.

$\square$

b) G is Eulerian iff every vertex of G has even degree (Thm 1.20, p.56)
iff n_1 and n_2 are both even, since every vertex in K_{n_1,n_2} are either of degree n_1 or degree n_2.

$\square$