Math 455: Discrete structures

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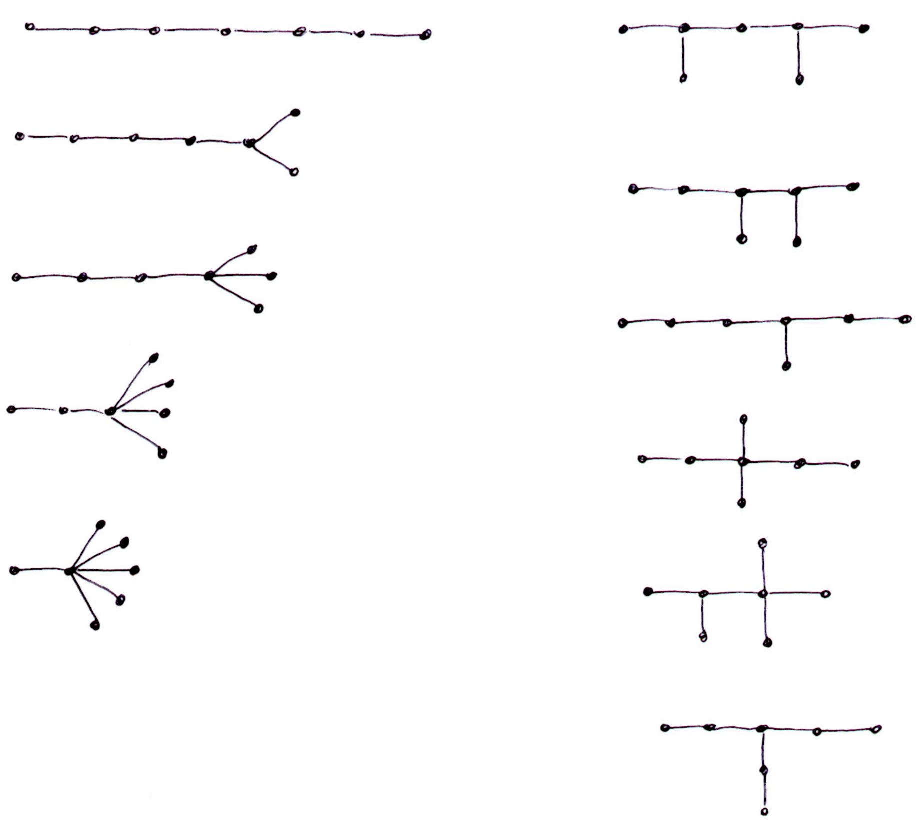
Homework 2 solutions

§1.3.1

1)



3) Let T be a tree of order $n \geq 2$. Let's choose any vertex v to be our starting point. Since T is a tree, it is a connected graph with no cycles. So for any other vertex $u \in T$ (which we know exists since $n \geq 2$), there exists a path from v to u, say $P(v,u)$. Moreover, this path is unique: if there were a different path from v to u, say $\tilde{P}(v,u)$, then taking $P(v,u)$ followed by $\tilde{P}(u,v)$ creates a cycle $\rightarrow\leftarrow$.

Now define two sets:

$X = \{ u \in V(G) \mid P(v,u) \text{ is of even length} \}$ and

$Y = \{ u \in V(G) \mid P(v,u) \text{ is of odd length} \}$.

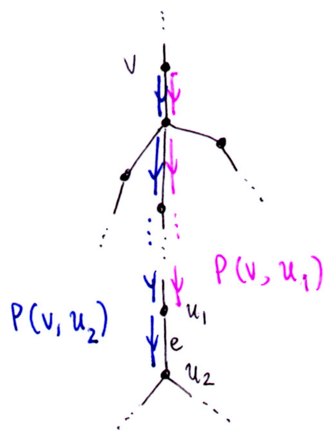
Then $X \cap Y = \emptyset$ and each vertex of T belongs to either X or Y. Fitting all vertices of T into the sets X and Y, we see that every edge in will be incident to one vertex in X and one vertex in Y:

§1.3.1

3) (cont-ed)

Suppose the end vertices of edge e are u_1 and u_2. Then there are paths $P(v, u_1)$ and $P(v, u_2)$:

example



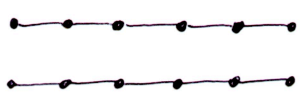
If the length of $P(v, u_1)$ is odd, then the length of $P(v, u_2)$ is even, i.e.,
if $u_1 \in Y$, then $u_2 \in X$, and
if the length of $P(v, u_1)$ is even, then the length of $P(v, u_2)$ is odd, i.e.,
if $u_1 \in X$, then $u_2 \in Y$.
Thus each edge in T will have one end vertex in X and the other in Y.

$\square$

§1.3.2

1 a) Impossible (too many edges)

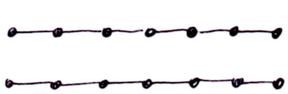
b)



c) Impossible

d)

e)



§1.3.2

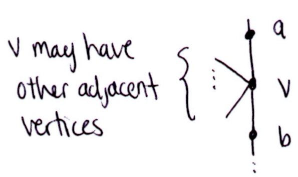
2) Suppose T has an even number of edges. Let's suppose that T has n vertices. Then by Thm 1.13 (p.35), T contains $n-1$ edges. So $n-1$ is even, and this forces n to be odd. By Thm 1.1 (p.6), the number of vertices with odd degree must be even. Then the maximum number of vertices with odd degree is $n-1$ (because n is odd). This leaves one vertex of even degree.
$\therefore T$ contains at least one vertex of even degree. $\square$

5) WTS: G is a tree if and only if for every pair of vertices u, v, there is exactly one path from u to v.
($\Rightarrow$) Suppose that G is a tree. Suppose by contradiction that there exists a pair of vertices u and v which are connected by two different paths, say

$$P_1 = u a_1 \cdots a_k v \quad \text{and} \quad P_2 = u b_1 \cdots b_\ell v$$

Then $u a_1 \cdots a_k v b_\ell \cdots b_1 u$ is a cycle in G, which is impossible since G is a tree $\rightarrow\leftarrow$ Thus each pair of vertices u, v must be connected by a unique path.
($\Leftarrow$) Suppose that for each pair of vertices u, v, there is exactly one path from u to v. Then G is connected. Moreover, G has no cycles: if there were a cycle in G, say $v a_1 a_2 \cdots a_i \cdots a_n v$, then $P_1 = v a_1 a_2 \cdots a_i$ and $P_2 = v a_n \cdots a_i$ are two different paths from vertex v to vertex a_i $\rightarrow\leftarrow$
Thus G is a connected, acyclic graph, so G is a tree. $\square$

8) WTS: every nonleaf in a tree is a cut vertex.
Let G be a tree and let v be a nonleaf. This means that $\deg(v) \geq 2$.
Now suppose that a and b are adjacent to v:



Then avb is the only path from a to b (it is unique by Exercise #5 above)

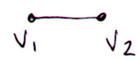
Then removal of vertex v will increase the number of components in G, so v is a cut vertex. $\square$

§1.3.2

10) Let T be a tree of order $n > 1$. We prove this by induction on n.

Basis step Suppose T has vertices. Then, since there are no vertices of degree 3, the number of leaves equals

$$2 = 2 + \sum_{\deg(v_i) \geq 3} (\deg(v_i) - 2)$$



Inductive step

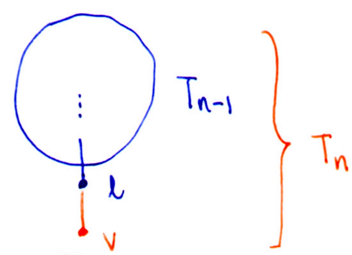
Inductive hypothesis Suppose that in a tree of order $n-1$, denoted T_{n-1}, the number of leaves equals

$$2 + \sum_{\deg(v_i) \geq 3 \text{ in } T_{n-1}} (\deg(v_i) - 2) \qquad \circledast$$

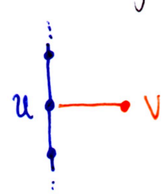
We WTS that $\circledast$ also gives the number of leaves in a tree of order n over vertices $v_i \in T_n$. A tree T_n of n vertices can be thought of as a tree T_{n-1} of $n-1$ vertices with one new vertex v added somewhere. We can only add a vertex v to T_{n-1} in the following ways:

(i) Add v to a leaf of T_{n-1}:

(say l is a leaf). Then in T_n, vertex v is a leaf, but l is not. So the number of leaves in T_n equals the number of leaves in T_{n-1}, given by $\circledast$ (done).



(ii) Add v to a degree 2 vertex in T_{n-1}:



Then in T_n, vertex v is now a leaf and vertex u has degree $2+1=3$.

So the number of leaves in T_n = (number of leaves in T_{n-1}) $\underbrace{+\ 1}_{\text{from vertex } v}$

$$= \underbrace{\left(2 + \sum_{\substack{\deg(v_i) \geq 3 \\ \text{in } T_{n-1}}} (\deg(v_i) - 2)\right)}_{\text{by inductive hypothesis}} + 1$$

§1.3.2

10) (cont.ed)

$$= \left(2 + \sum_{\substack{deg(v_i)\geq 3 \\ \text{in } T_{n-1}}} (deg(v_i) - 2)\right) + \underbrace{(deg(u) - 2)}$$

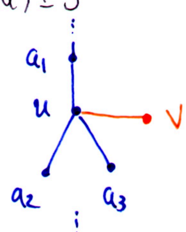
Since $deg(u) - 2 = 3 - 2 = 1$ in T_n

Notice! The degrees of these vertices v_i remain unchanged in T_n after adding u; so we may rewrite this as "$deg(v_i) \geq 3$ in T_n"

$$= \underbrace{\left(2 + \sum_{\substack{deg(v_i)\geq 3 \\ \text{in } T_n}} (deg(v_i) - 2)\right)}$$ which is ✱ (done)

Now this sum includes $deg(u) - 2$ since $deg(u) \geq 3$

(iii) Add v to a vertex of T_{n-1} with degree ≥ 3:
(Similar to the case above)



Number of leaves in T_n = (number of leaves in T_{n-1}) + 1

$$= \left(2 + \sum_{\substack{deg(v_i)\geq 3 \\ \text{in } T_{n-1},\ v_i \neq u}} (deg(v_i) - 2)\right) + \left((deg(u) + 1) - 2\right) + 1$$

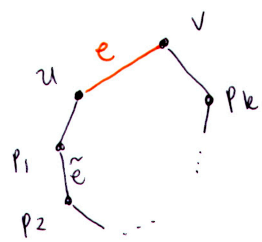
$$= 2 + \sum_{\substack{deg(v_i)\geq 3 \\ \text{in } T_n}} (deg(v_i) - 2)$$ because in T_n, vertex u has degree ≥ 3

§1.3.3

2) WTS: A graph is a tree if and only if it is connected and has exactly one spanning tree.

($\Rightarrow$) Let G be a tree. Then G is connected by definition. Also G is a spanning tree of itself. Since any two vertices $u, v \in G$ are connected by a unique path, G must be the unique spanning tree.

($\Leftarrow$) Let G be a connected graph with a unique spanning tree, say T. Suppose by contradiction that G is not a tree. Then G contains an edge e that is not in T. Since T is a spanning tree of G, there exists a path from u to v in T, say $P = u p_1 p_2 \cdots p_k v$. Now if we add edge e to path P, we have a cycle:



Next we claim that we can form another spanning tree $\tilde{T}$ of G:

Let $\tilde{T} = T + e - \tilde{e}$, where $\tilde{e}$ is any edge in path P; let's suppose $\tilde{e}$ is the edge adjacent to p_1 and p_2.

Now $\tilde{T}$ is a tree because it's connected and contains no cycles; by construction it contains all vertices of G and is different from T since $\tilde{e} \notin \tilde{T}$ but $\tilde{e} \in T$.

So we have two distinct spanning trees of G, contradicting our assumption $\rightarrow\leftarrow$

Thus G is a tree.

$\square$

4) Let G be connected and e an edge of G. WTS: e is a bridge if and only if it is in every spanning tree of G.

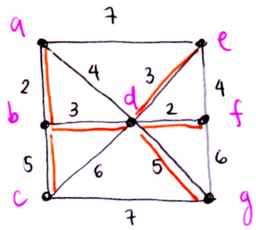
($\Rightarrow$) Suppose that e is a bridge. Say it is incident to vertices u and v. Let T be any spanning tree of G. So T includes all vertices of G. But e is a bridge means that e is the only edge in G connecting vertices u and v, so e must be in T.

($\Leftarrow$) Suppose that e is not a bridge. This means that, if u, v are the end vertices of e, then there is a path from u to v not containing e, so there is a spanning tree of G that does not contain e (contrapositive).

$\square$

§1.3.3

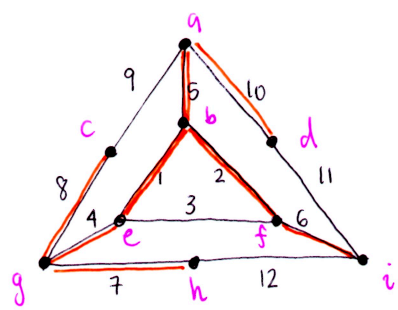
5)



(One possibility)

In order:
(edge , weight)

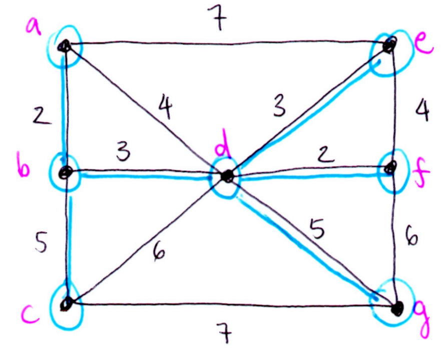
({a, b}, 2)
({d, f}, 2)
({b, d}, 3)
({d, e}, 3)
({b, c}, 5)
({d, g}, 5)



({b, c}, 1)
({b, f}, 2)
({g, e}, 4)
({a, b}, 5)
({f, i}, 6)
({g, h}, 7)
({g, c}, 8)
({a, d}, 10)

§1.3.3

6)



① Choose vertex a

$(\{a,b\},2)$ ← min weight

$(\{a,d\},4)$

$(\{a,e\},7)$

② Choose vertex b

$(\{b,d\},3)$ ← min

$(\{b,c\},5)$

③ Choose vertex d

min: $(\{d,f\},2)$

④ Choose vertex f

min: $(\{d,e\},3)$

⑤ Choose vertex e

min: $(\{d,g\},5)$

⑥ choose vertex g

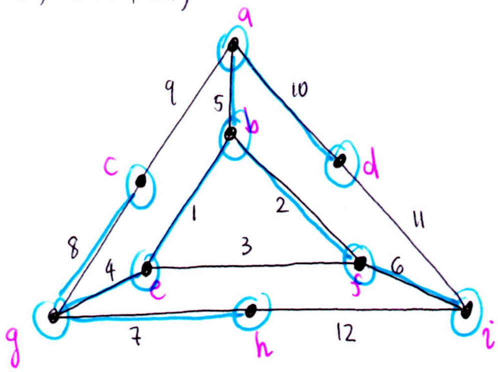
min: $(\{b,c\},5)$

⑦ choose vertex c

Now every vertex in G has been chosen, so we end here.

§1.3.3

6) (cont-ed)



① vertex a
({a,b},5) ← min

② vertex b
({b,e},1)

③ vertex e
({b,f},2)

④ vertex f
({e,g},4)

⑤ vertex g
({f,i},6)

⑥ vertex i
({g,h},7)

⑦ vertex h
({g,c},8)

⑧ vertex c
({a,d},10)

⑨ vertex d

End here