Math 455 : Discrete Structures

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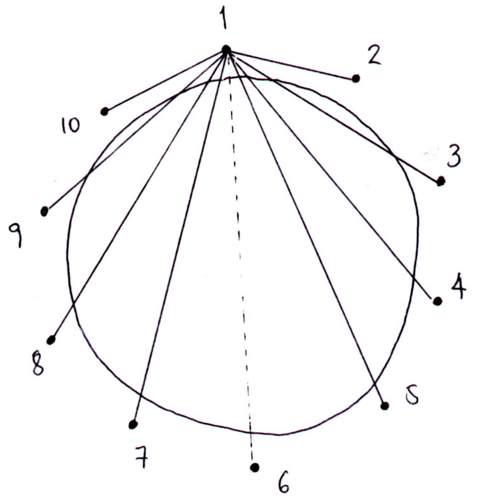
Homework 1 Solutions

§1.1.1

1) Handshake diagram for person #1:

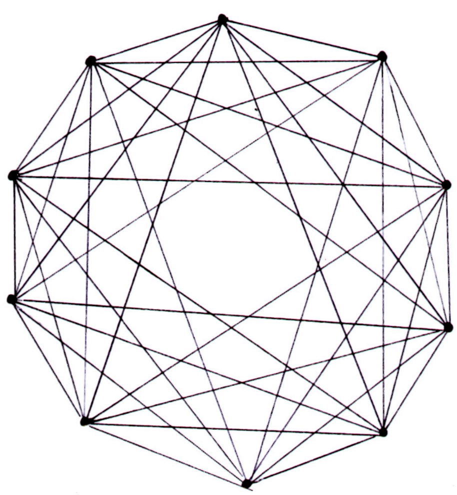


• = person

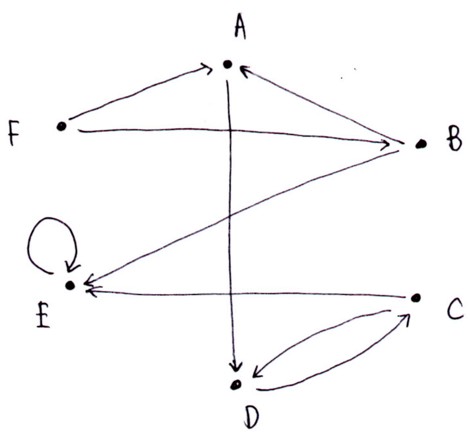
— = shake hands

---- = do not shake hands

So the graph that models this situation looks like:



2)



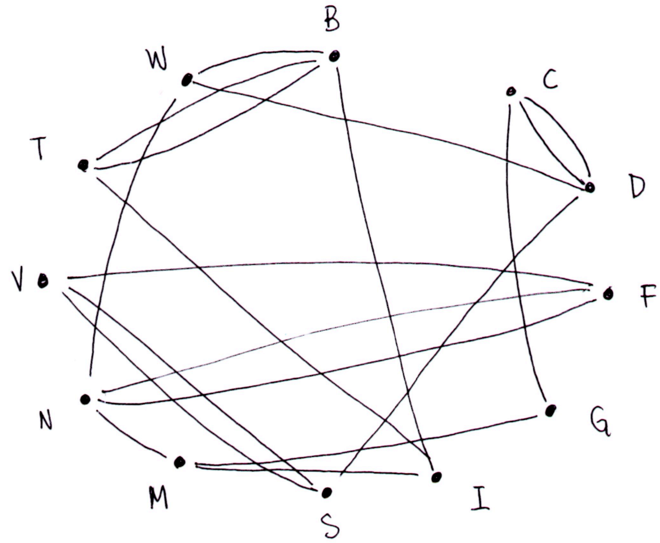
A = Adam

B = Bert

⋮

§1.1.1

8).



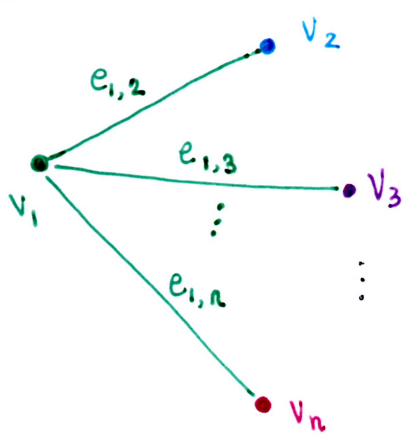
§1.1.2

1) If G is a graph of order n, then the maximum number of edges in G is $\frac{n(n-1)}{2}$

Reason:

Suppose we label the vertices of G as $v_1, v_2, \dots, v_n$.
Then the maximum number of edges incident to v_1 is $n-1$, each edge connecting v_1 to another v_i where $i = \underbrace{2, \dots, n}_{n-1 \text{ possibilities}}$

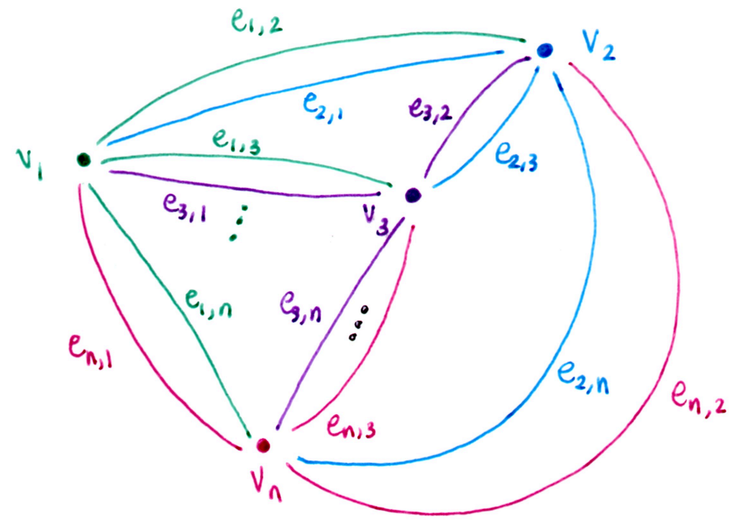
Let's denote $e_{1,i}$ to be the edge connecting vertex v_1 to vertex v_i, for $i = 2, \dots, n$.
Notice that $i \neq 1$, because otherwise G would have a loop at v_1 (we are assuming G has no loops and no multiple edges).
So we may picture this as:



§1.1.2

1) (conted)

· If we do this for all n vertices, since each vertex is incident to $n-1$ edges, we have a total of $n(n-1)$ edges.
But we are double counting the edges! A picture:



Remember: G does not have multiple edges

Notice that edges $e_{i,j}$ and $e_{j,i}$ connect the same vertices for every $i,j \in \{1,\dots,n\}$, $i \neq j$.
So cutting $n(n-1)$ in half will get rid of all multiple edges.
$\therefore$ the maximum number of edges in G is $n(n-1)/2$.

$\square$

3 a) A path is a walk with <u>distinct</u> vertices.
The length of a walk equals the number of its edges.
For example, the walk $a - b - c - d - e$ is of length 4.
Let's denote W_i to be any walk in G of length i that ends with c. There are 5 vertices in G, so the maximum walk is of length 4.
Let $x, y, z, w \in \{a, b, d, e\}$ be distinct vertices. Then,

- A walk W_4 looks like

$$x - y - z - w - c$$

There are 4 choices for x, 3 for y, 2 for z, and 1 for w.
This gives $4 \cdot 3 \cdot 2 \cdot 1 = 24$ choices for W_4

- A W_3 looks like

$$x - y - z - c$$

$\Rightarrow$ there are $4 \cdot 3 \cdot 2 = 24$ choices for W_3

§1.1.2

3 a) (cont.ed)

- A walk W_2:

$$x - y - c$$

$\Rightarrow 4 \cdot 3 = 12$ choices for W_2

- W_1:

$$x - c$$

$\Rightarrow 4$ choices for W_1

$\therefore 24 + 24 + 12 + 4 = 64$ different paths in G end with vertex c.

b) Using the same notation as part (a):

Now we want to avoid vertex c altogether, so $x, y, z, w \in \{a, b, d, e\}$.

Notice we now cannot have a walk of length 4.

- For W_3:

$$x - y - z - w$$

$\Rightarrow 4 \cdot 3 \cdot 2 \cdot 1 = 24$ choices

- For W_2:

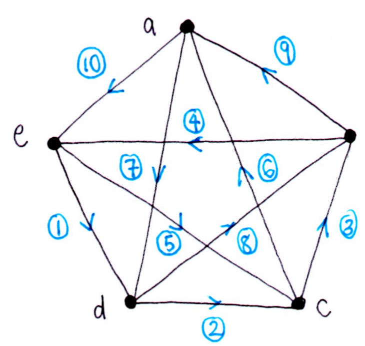
$\Rightarrow 4 \cdot 3 \cdot 2 = 24$ choices

- For W_1:

$\Rightarrow 4 \cdot 3 = 12$ choices

$\therefore 24 + 24 + 12 = 60$ different paths in G avoid vertex c.

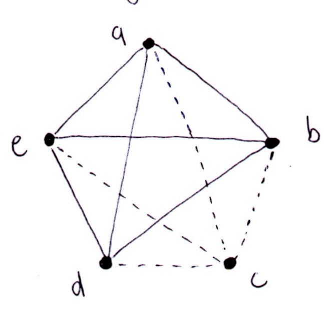
c) The maximum circuit length is 10; Example:



e-d-c-b-e-c-a-d-b-a-e

d) The maximum length of a circuit that does not include vertex c is 4.

Excluding vertex c gives the complete graph on 4 vertices, K_4:



(Some edges disappear when we exclude c; these are dotted)

§1.1.2

3d (cont.ed)

We cannot get a circuit of length 5, but there are circuits of length 4. (Example: a — b — d — e — a).

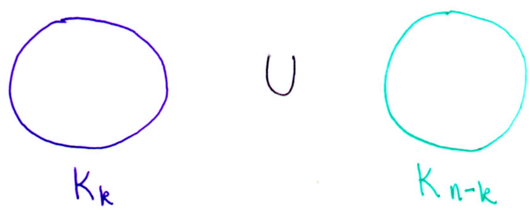
$\square$

9) The maximum size of G is $\dfrac{(n-1)(n-2)}{2}$.

Pf

Suppose that G is a disconnected graph of n vertices. Then there are two connected components of G. In order to maximize edges, we want each component to be a complete graph. Suppose that one component has k vertices and the other as $n-k$:

$G =$



where $1 \le k \le n-1$ (so the components are nonempty)

In K_k there are $\dfrac{k(k-1)}{2}$ edges and in K_{n-k} there are $\dfrac{(n-k)(n-k-1)}{2}$ edges.

$$\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = \frac{k^2-k}{2} + \frac{(n-k)^2-(n-k)}{2}$$

$$= \frac{n^2-(2k+1)n+2k^2}{2}, \quad 1 \le k \le n-1$$

If $k=1$:

$$\frac{n^2-(2+1)n+2}{2} = \frac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}$$

If $k=n-1$:

$$\frac{n^2-[2(n-1)+1]n+2(n-1)^2}{2} = \frac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}$$

§1.1.2

11) An edge e is a bridge if and only if e lies on no cycle of G.

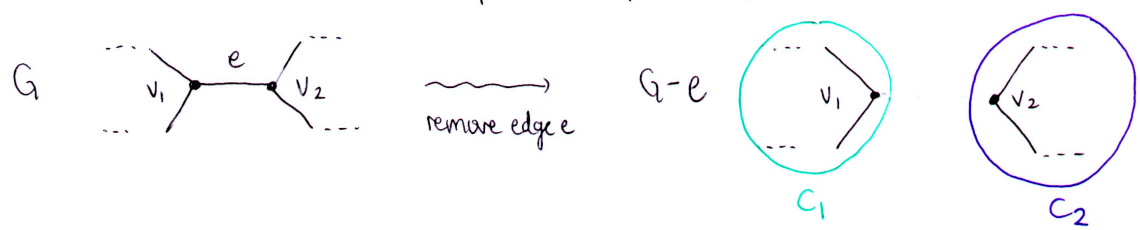
Pf

($\Rightarrow$) WTS: if e is a bridge of G, then e lies on no cycle of G.

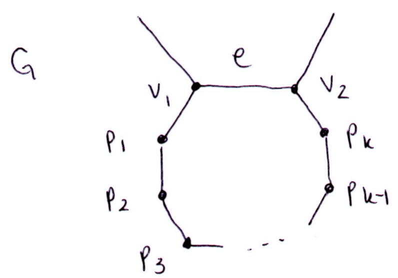
$\Updownarrow$ (equivalent to)

If e does lie on a cycle of G, then e is not a bridge of G (contrapositive).

* Suppose the contrapositive is false, i.e., suppose e lies on a cycle of G, but is also a bridge.
Since e is a bridge, in $G-e$, the two vertices incident to e (say v_1 and v_2) will lie in different connected components C_1 and C_2:



Then there is no path from v_1 to v_2 in $G-e$.
We also assumed that e lies on a cycle of G. Then v_1 and v_2 must also lie on this cycle, and we may write this cycle as $v_1 p_1 \cdots p_k v_2 v_1$, where $p_1, \ldots, p_k$ are some vertices in G and $k \geq 1$. Picture:



But in $G-e$, the vertices v_1 and v_2 do not lie in different components because the path $v_1 p_1 \cdots p_k v_2$ will still connect them, a contradiction $\rightarrow\leftarrow$
Then our assumption * was incorrect, so the contrapositive is true.

($\Leftarrow$) WTS: If e lies on no cycle of G, then e is a bridge of G.
Suppose e lies on no cycle of G. Suppose that e is incident to v_1 and v_2.
Then there is no path from v_1 to v_2 which does not include edge e.
So in $G-e$, there is no path from v_1 to v_2, so v_1 and v_2 lie in different components. Since removal of e breaks G into more components, edge e is a bridge.

$\square$

§1.1.3

2) Suppose that K_{r_1,r_2} is regular.

K_{r_1,r_2} is a complete bipartite graph with partite sets X and Y, where $|X| = r_1$ and $|Y| = r_2$.
If a vertex $v \in K_{r_1,r_2}$, then either $v \in X$ or $v \in Y$.

If $v \in X$, then it is adjacent to each edge in Y, so $\deg(v) = r_2$.

If $v \in Y$, then it is adjacent to each edge in X, so $\deg(v) = r_1$

But K_{r_1,r_2} is regular, so every vertex has the same degree.

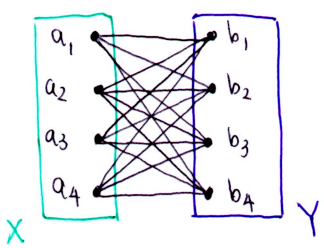
$\therefore r_1 = r_2$.

$\square$

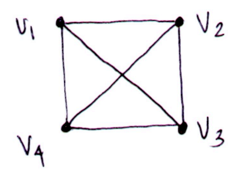
3) No, K_4 is not a subgraph of $K_{4,4}$.

Let's label the graphs $K_{4,4}$ and K_4 in the following way:

$K_{4,4}$



K_4



Suppose that K_4 were a subgraph of $K_{4,4}$. This means that

$$\underbrace{V(K_4)}_{\text{vertex set of } K_4} \subseteq V(K_{4,4})$$

and

$$\underbrace{E(K_4)}_{\text{edge set of } K_4} \subseteq E(K_{4,4})$$

WLOG let's start with vertex $v_1 \in K_4$. Since v_1 is adjacent to v_2, v_3, and v_4, we must have $v_1 \in X$ and $v_2, v_3, v_4 \in Y$. But v_2 and v_3 are adjacent, so this is a contradiction (vertices in Y are not adjacent to each other). $\rightarrow\leftarrow$

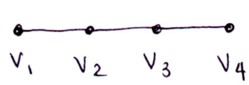
$\therefore K_4$ is not a subgraph of $K_{4,4}$.

$\square$

4) No, P_4 is not an induced subgraph of $K_{4,4}$.

Let's suppose by contradiction that P_4 is an induced subgraph of $K_{4,4}$.
Say we label $K_{4,4}$ the same way as in #3 above, and say we label P_4 as

P_4



§ 1.1.3

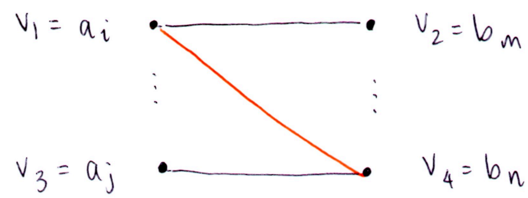
4) (Cont.ed)

Then $V(P_4) \subseteq V(K_{4,4})$, and $E(P_4) = \{u,v \mid v \in V(P_4) \text{ and } uv \in E(K_{4,4})\}$ by definition of induced subgraph.

Since v_1 and v_2 are adjacent in P_4, if we consider them as vertices in $K_{4,4}$, they must belong to different partite sets; WLOG suppose $v_1 \in X$ and $v_2 \in Y$.

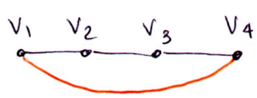
Similarly since v_2 and v_3 are adjacent, we must have $v_3 \in X$, and $v_4 \in Y$.

Now let's suppose that $v_1 = a_i$ and $v_3 = a_j$ and $v_2 = b_m$ and $v_4 = b_n$ where $1 \leq i, j \leq 4$ and $i \neq j$, and $1 \leq m, n \leq 4$ and $m \neq n$.



By definition of induced subgraph, P_4 must contain all edges between v_1, v_2, v_3, and v_4 that appear in $K_{4,4}$.

So the orange edge between v_1 and v_4 must lie in P_4. But then P_4 would look like



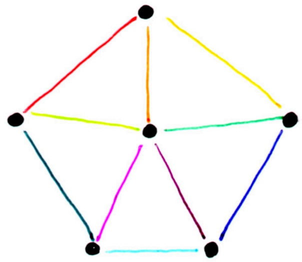
$\neq P_4$ $\rightarrow\leftarrow$

$\therefore P_4$ is not an induced subgraph of $K_{4,4}$.

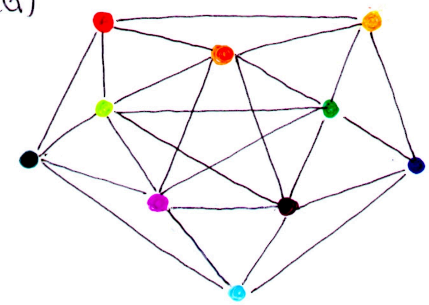
$\square$

§1.1.3

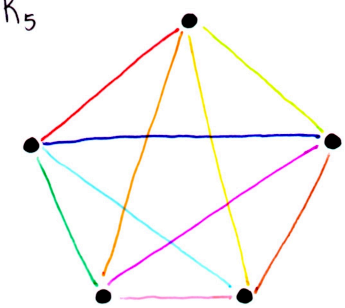
7a) G



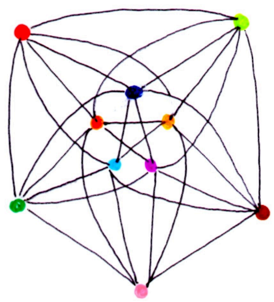
L(G)



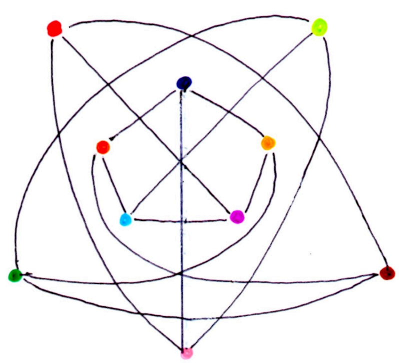
b) K_5



$L(K_5)$



$L(K_5)^c$



§1.1.3

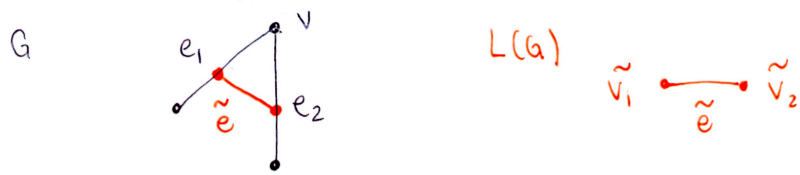
7 c) Suppose that G has n vertices, labeled $v_1, \ldots, v_n$, and $\deg(v_i) = r_i$.
Let m denote the size of G, so $r_1 + r_2 + \cdots + r_n = 2m$.
Then (i) the size of $L(G)$ is

$$\sum_{i=1}^{n} \binom{r_i}{2} = \sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}$$

(ii) the order of $L(G) = m$

(ii) follows from the fact that each edge in G becomes a vertex in $L(G)$, so that the m edges in G give m vertices in $L(G)$

(i) An edge $\tilde{e} \in L(G)$ mes from two edges $e_1, e_2 \in G$ that are adjacent to the same vertex $v \in G$:

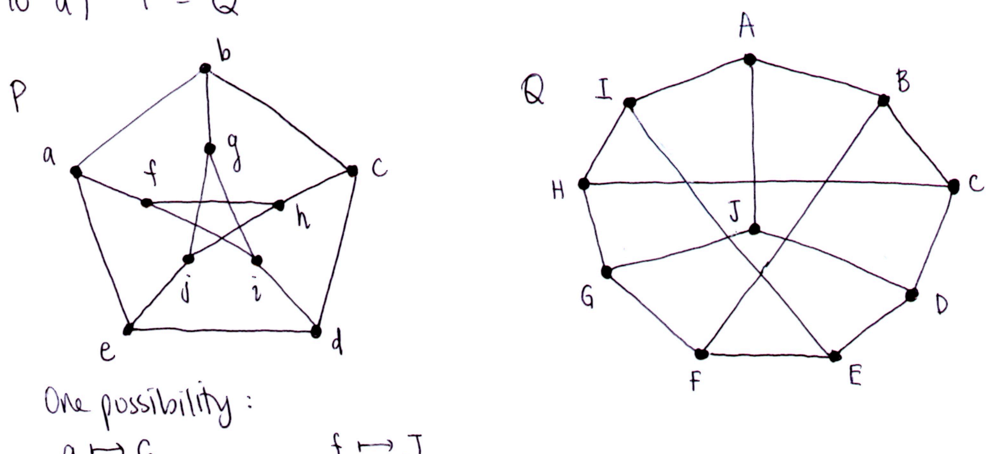


We note that the vertex v is uniquely determined by edges e_1 and e_2.
If $\deg(v) = d$, then there are a total of $\binom{d}{2}$ pairs of edges $\{e_i, e_j\}$ such that e_i and e_j are adjacent to v. Since G has n vertices, the total number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \binom{r_i}{2} = \sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}$$

□

10 a) $P \cong Q$



One possibility :

$a \mapsto G$ $\quad$ $f \mapsto J$
$b \mapsto H$ $\quad$ $g \mapsto I$
$c \mapsto C$ $\quad$ $h \mapsto D$
$d \mapsto B$ $\quad$ $i \mapsto A$
$e \mapsto F$ $\quad$ $j \mapsto E$

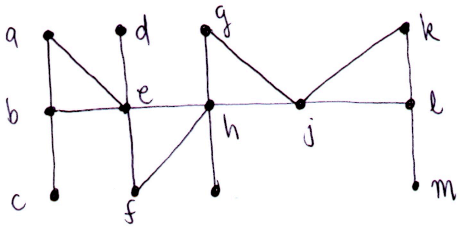
§1.1.3

10 b) R has a cycle of 4 vertices, but P and Q do not.

□

§1.2.1

1)



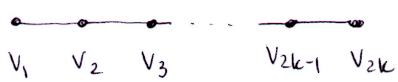
radius = 3

diameter = 6

center = $\{h\}$

Eccentricity	Example
ecc(a) = 5	aehjlm
ecc(b) = 5	behjlm
ecc(c) = 6	cbehjlm
ecc(d) = 5	dehjlm
ecc(e) = 4	ehjlm
ecc(f) = 4	fhjlm
ecc(g) = 4	ghjlm
ecc(h) = 3	hjlm
ecc(i) = 4	ihjlm
ecc(j) = 4	jhebc
ecc(k) = 5	kjheba
ecc(l) = 5	ljheba
ecc(m) = 5	mljhea

2) $\boxed{P_{2k}}$:



$ecc(v_1) = 2k-1 = ecc(v_{2k})$

$ecc(v_2) = 2k-2 = ecc(v_{2k-1})$

⋮ decreases

Since the greatest eccentricity is $2k-1$, we have $diam(P_{2k}) = 2k-1$.
The smallest ecentricity is $2k/2 = k$

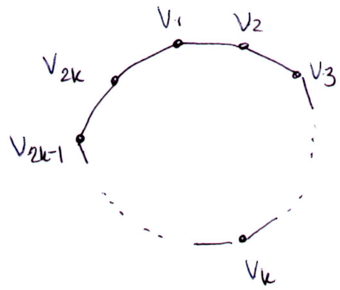
$\Rightarrow rad(P_{2k}) = k$

$\boxed{P_{2k-1}}$ similar reasoning as P_{2k}:

$rad(P_{2k+1}) = k$, $diam(P_{2k}) = 2k$

§1.2.1

2) $\boxed{C_{2k}}$



$ecc(v_1) = ecc(v_2) = \cdots = ecc(v_{2k}) = k$

$\Rightarrow rad(C_{2k}) = k = diam(C_{2k})$

$\boxed{C_{2k+1}}$ Similar to C_{2k}:

$rad(C_{2k+1}) = k = diam(C_{2k+1})$

$\boxed{K_n}$ Every vertex is adjacent to every other vertex

$\Rightarrow rad(K_n) = 1 = diam(K_n)$

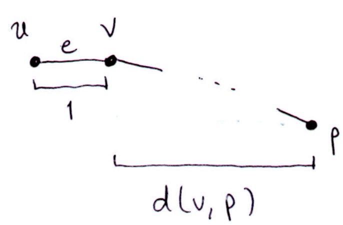
$\boxed{K_{m,n}}$ $rad(K_{m,n}) = 1 = diam(K_{m,n})$

5) Suppose that u and v are adjacent vertices in a graph.

WLOG suppose $ecc(v) \leq ecc(u)$. Let p be another vertex in the graph s.t. $p \neq v$. ("such that")

Then the maximal distance from u to v is :

$$1 + \max_{x \in V(G)} d(p,v)$$



Then $ecc(v) = \max_{x \in V(G)} d(p,v)$

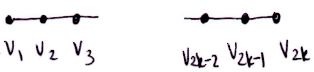
$= 1 + \max_{x \in V(G)} d(p,u)$

$= ecc(u)$

$\square$

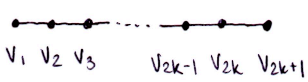
§1.2.2

1 a) P_{2k}:



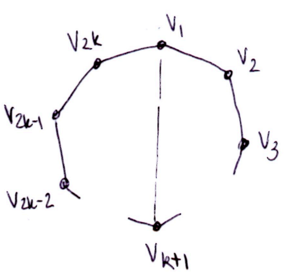
$$\begin{array}{cc} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_{2k-2} & v_{2k-1} & v_{2k} \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{2k-1} \\ v_{2k} \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & \vdots & & \\ 0 & 0 & 0 & 0 & & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 \end{bmatrix} \end{array}$$

P_{2k+1} :



$$\begin{array}{cc} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_{2k-2} & v_{2k-1} & v_{2k} & v_{2k+1} \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2k-2} \\ v_{2k-1} \\ v_{2k} \\ v_{2k+1} \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & \vdots & & & \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

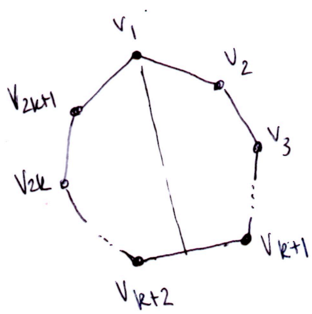
C_{2k} :



$$\begin{array}{cc} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_{2k-2} & v_{2k-1} & v_{2k} \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2k-2} \\ v_{2k-1} \\ v_{2k} \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 1 & 0 \end{bmatrix} \end{array}$$

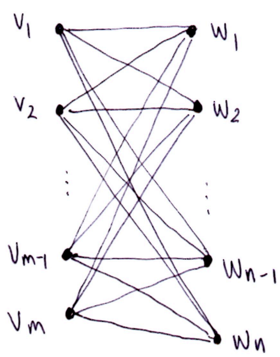
§1.2.2

1 b) C_{2k+1}:



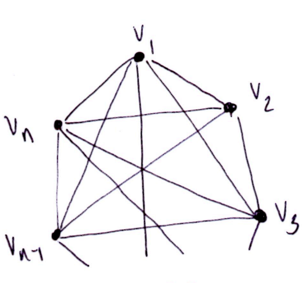
$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{2k-2} & v_{2k-1} & v_{2k} & v_{2k+1} \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2k-2} \\ v_{2k-1} \\ v_{2k} \\ v_{2k+1} \end{array} &
\left[\begin{array}{ccccccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & & & \ddots & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

c) $K_{m,n}$:



$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} v_1 & v_2 & \cdots & v_{m-1} & v_m & w_1 & w_2 & \cdots & w_{n-1} & w_n \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{m-1} \\ v_m \\ w_1 \\ w_2 \\ \vdots \\ w_{n-1} \\ w_n \end{array} &
\left[\begin{array}{cccccccccc}
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & & \ddots & & \vdots & \vdots & & \ddots & & \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & \ddots & \vdots & & \vdots & & & & \\
1 & 1 & & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & & 1 & 1 & 0 & 0 & \cdots & 0 & 0
\end{array}\right]
\end{array}
$$

d) K_n:



$$
\begin{array}{cc}
 & \begin{array}{cccccc} v_1 & v_2 & v_3 & \cdots & v_{n-1} & v_n \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{array} &
\left[\begin{array}{cccccc}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & & 1 & 1 \\
\vdots & & & \ddots & 1 & 1 \\
1 & 1 & 1 & \cdots & 0 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{array}\right]
\end{array}
$$

§1.2.2

3) Let's label entries in A in the following way: (suppose G has n vertices $v_1, \ldots, v_n$)

$$A = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & \cdots & v_j & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_j \\ \vdots \\ v_n \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

where $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$, for $1 \leq i, j \leq n$.

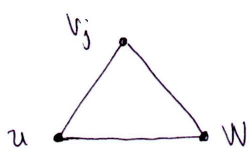
By the rules of matrix multiplication, the (j,j)th entry of A^2 is

$$a_{j1}a_{1j} + a_{j2}a_{2j} + \cdots + a_{jj}^2 + \cdots + a_{nj}a_{jn} \quad *$$

Note that $a_{ij} = a_{ji}$, and $a_{ij}a_{ji} = 1$ iff vertices v_i and v_j share an edge. Then $*$ is just counting the number of vertices that share an edge with vertex v_j, or, the degree of v_j.

$\square$

4 a) By Theorem 1.7 (p 23), the value of entry $[A^3]_{j,j}$ gives the number of 3-walks from v_j back to v_j, or, the number of 3-cycles at v_j. But we are doble counting. For example, the walks below are the same:



$$v_j \to w \to u \to v_j$$
$$v_j \to u \to w \to v_j$$

So we divide by $\frac{1}{2}$ to obtain $\frac{1}{2}[A^3]_{j,j}$.

b) By a), the number of triangles that contain v_j is $\frac{1}{2}[A^3]_{j,j}$. So to get the number of triangles containing $v_1, v_2, \ldots, v_n$, we add

$$\frac{1}{2}[A^3]_{1,1} + \frac{1}{2}[A^3]_{2,2} + \cdots + \frac{1}{2}[A^3]_{n,n}$$

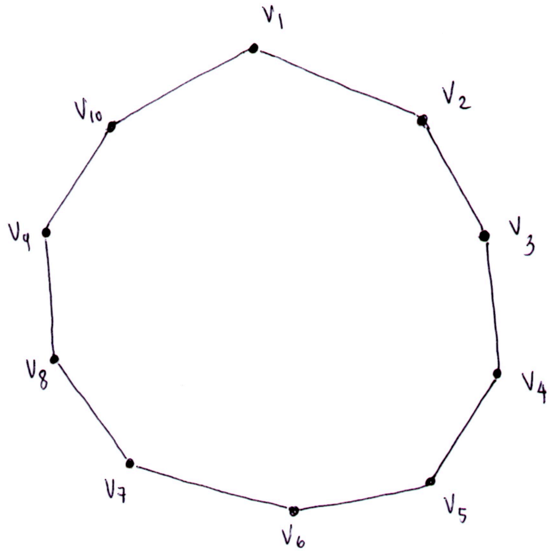
$$= \sum_{i=1}^{n} \frac{1}{2}[A^3]_{i,i}$$

$$= \frac{1}{2}\operatorname{Tr}(A^3).$$

However, we are counting each triangle three times (once for each vertex), so the number of triangles in G is $\frac{1}{3} \cdot \frac{1}{2}\operatorname{Tr}(A^3) = \frac{1}{6}\operatorname{Tr}(A^3)$. $\square$

§1.2.2

5) C_{10}:



☆ Claim If we start at v_1, take a walk of length n, and land on v_i for some $1 \le i \le 10$, then

$$i \text{ is } \begin{cases} \text{an even number} & \text{if } n \text{ is odd} \quad (1) \\ \text{an odd number} & \text{if } n \text{ is even} \quad (2) \end{cases}$$

We can prove this by induction.

Base case ($n=1$) If we start at v_1 and take a 1-walk, then we land on v_2 (so $i=2$, an even number). So the claim is true for $n=1$.

Inductive step

Inductive hypothesis Suppose that the claim holds for $n-1$. That is, if we start at v_1, take a walk of length $n-1$, and land on v_i for some $1 \le i \le 10$, then

$$i \text{ is } \begin{cases} \text{an even number} & \text{if } n-1 \text{ is odd} \\ \text{an odd number} & \text{if } n-1 \text{ is even.} \end{cases}$$

We want to show the claim is true for n, i.e., that ☆ is true.

Let's start at v_1 and take an $(n-1)$-walk, where n is even. Say we land on v_i. Since n is even, we know that $n-1$ is odd, and by the inductive hypothesis, i is an even number. Thus, for even n-walk, we get i is odd, proving (1).

going from an $(n-1)$ walk to n walk is one more "step", so i changes parity from even to odd

Now let's start at v_1, and take an $(n-1)$-walk, where n is odd. Then by inductive hypothesis, since $n-1$ is even, the i is odd. Thus for an odd n-walk, we get i is even, proving (2).

∴ ☆ is true.

Now to answer the question:

The $(1,5)$ entry of A^{2009} is the number of (2009)-walks from vertex v_1 to v_5.

So here, we have $n = 2009$, an odd number. But starting at v_1 and landing on v_5 tells us that $i=5$, an odd number. By the Claim, this is impossible, so the entry of $(1,5)$ in A^{2009} is 0. □