Reliability polynomial etc

Will assign HW due after break.

Chromatic polynomial.

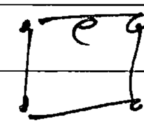
G graph.

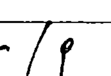
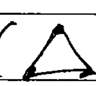
$C_G(k) = \# \{ k\text{-colourings of } G \}$.
poly in k.

k-colouring means a function $V(G) \to \{1, .., k\}$.
s.t. $f(v) \neq f(w)$ if $\{v, w\} \in E(G)$.

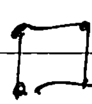
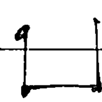
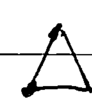
Thm $C_G(k) = C_{G-e}(k) - C_{G/e}(k)$
$e \in E(G)$
$G - e$ "delete e"
G/e "contract e, remove any mult. edges that result"

Thm $\Rightarrow C_G(k)$ is poly.

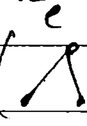
e.g. $C_4 = G$. 

shorthand for "the poly" $\Rightarrow C($  $) = C($  $) - C($  $)$

abbreviate more...

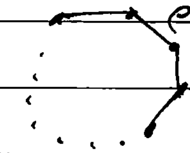
 $=$  $-$ 

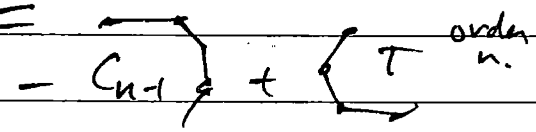
$= k(k-1)^3 -$ 

$= k(k-1)^3 - ($  $-$  $)$

Previous class:
$C_T(k) = k(k-1)^{n-1}$
$n = |T|$

$$= k(k-1)^3 - k(k-1)^2 + k(k-1)$$

general cycle: $C_{C_n}(k) =$ 

$=$ 

$$\Rightarrow C_{C_n}(k) = k(k-1)^{n-1} - C_{C_{n-1}}(k)$$

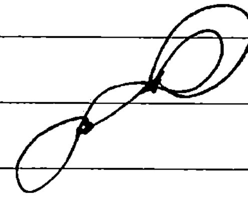
Other polynomials attached to graphs.

- Reliability polynomial $R_G(p)$
- Tutte polynomial $T_G(x,y)$

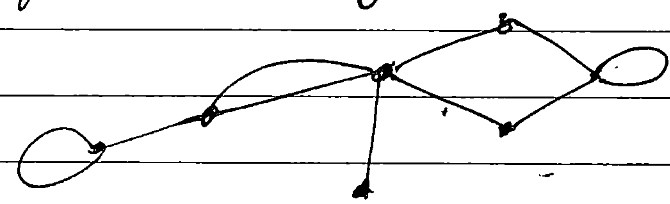
Both chromatic and reliability polys are specializations of $T_G(x,y)$

Reliability polynomial.

For the moment (for these topics) we allow loops
multiple edges

in graphs. 

G graph. say G models a network.



verts computers
edges connections

take $0 \le p \le 1$ probability.
interpret it as probability that any given edge will fail.

goal: compute probability that the network "survives" (i.e. remains connected) if all edges fail independently with probability p.

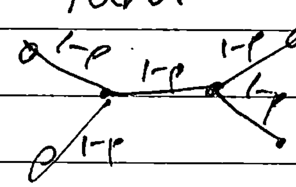
e.g. G: 

edge fails	p
edge survives	$1-p$

$$R_G(p) = 1-p$$

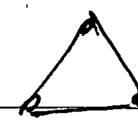
surv.

e.g. G: tree



$$R_G(p) = (1-p)^5$$

in general: $G = T \Rightarrow R_T(p) = (1-p)^{n-1}$
$|T| = n$

e.g. $G = K_3$ 

prob of survival $= P(\{\text{all 3 survive}\})$ ⊗ $+ P(\{\text{exactly 2 survive}\})$ ⊙

$$= \overset{⊗}{(1-p)^3} + \overset{⊙}{3p(1-p)^2}$$

HTT
THT
TTH

$$= (1-p)^2(1-p+3p)$$

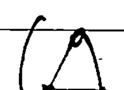
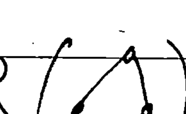
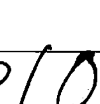
$$= (1-p)^2(1+2p).$$

Like the chromatic poly, $R_G(p)$ satisfies a deletion – contraction recursion relation.

$$R_G(p) = \boxed{X}\, R_{G-e} + (X)\, R_{G/e}$$

<u>Thm</u> $R_G(p) = \begin{cases} ⓐ\ R_{G-e}(p) & \text{if } e \text{ loop} \\ ⓑ\ (1-p)R_{G/e}(p) & \text{if } e \text{ bridge} \\ ⓒ\ pR_{G-e}(p) + (1-p)R_{G/e} & \text{otherwise} \end{cases}$

e.g. $G = K_3$ $\quad R_G(p)$?

$R($  $) \overset{ⓒ}{=} p\,R($  $) + (1-p)\,R($  $)$

$$= p\underset{ⓑ}{(1-p)^2} + \cdots$$

$$= p(1-p)^2 + (1-p)\big(pR(\;) + (1-p)R(\;)\big)$$

b a

$$= p(1-p)^2 + (1-p)\big(p(1-p) + (1-p)\big)$$

$$= p(1-p)^2 + p(1-p)^2 + (1-p)^2$$

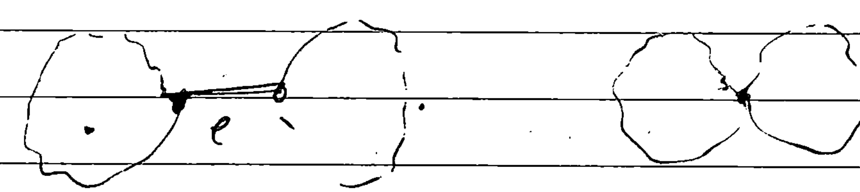
$$= (1-p)^2(1+2p) \checkmark$$

Why is this true?

(a) $R_G(p) = R_{G-e}(p)$ if e loop
pretty clear.

(b) $R_G(p) = (1-p)R_{G/e}$ if e bridge

G



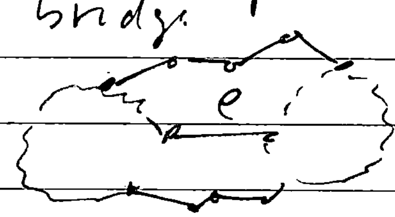
$$(1-p)R_{G/e}(p)$$

prob that e survives

prob. of survival even with failures away from e.

(c) e neither loop nor bridge

G:



prob. of network survival:

2 cases:

$$R_G(p) = \underbrace{\{\text{Pr of e disappearing}\} \times \{\text{Pr of survival with e gone}\}}_{\text{e gone}} + \underbrace{\{\text{pr of e remaining}\} \times \{\text{pr of survival with e there}\}}_{\text{e stays.}}$$

$$= p\, R_{G-e}(p) + (1-p)\, R_{G/e}(p)$$

Variations: different probabilities for different edges.