LINEAR RECURRENCES AND RATIONAL GENERATING FUNCTIONS

Let $f(x) = \sum_k a_k x^k$ be a generating function. We say the sequence $a_0, a_1, \ldots$ satisfies a *linear recurrence relation of order* n if there exist constants $c_1, \ldots, c_n$ such that

$$a_k = c_1 a_{k-1} + \cdots + c_n a_{k-n} \tag{1}$$

for all $k \geq N$, where N is some fixed number $\geq n$. The basic example is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, which is defined by $a_0 = a_1 = 1$ and $a_k = a_{k-1} + a_{k-2}$ for $k \geq 2$.

We have the following fact about the generating functions of sequences defined by recurrences:

Theorem 1. *Suppose the coefficients of $f(x) = \sum a_k x^k$ satisfy (1) for all $k \geq N$, where $N \geq n$ is given. Suppose further that the values a_i for $i < N$ are known. Then*

$$f(x) = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_n x^n}, \tag{2}$$

where $P(x) = A_0 + A_1 x + \ldots A_{N-1} x^{N-1}$ is a polynomial of degree at most $N - 1$.

The coefficients of the numerator $P(x)$ are determined by the a_i, $i < N$, but in general $A_i \neq a_i$. A function of the form polynomial divided by polynomial is called a *rational function*. Sometimes people summarize the conclusion of the theorem by saying that the generating function of a sequence defined by a linear recurrence is a rational function.

Consider the Fibonacci example. According to the theorem the generating function for them can be written as

$$\frac{A_0 + A_1 x}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots \tag{3}$$

We can find the A_i by multiplying both sides of (3) by $1 - x - x^2$. There is a lot of cancellation (that's in fact the whole point of how this works), and we find

$$\begin{aligned} A_0 + A_1 x &= (1 - x - x^2)(1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots) \\ &= 1 + 0x + 0x^2 + 0x^3 + \cdots, \end{aligned}$$

which means $A_0 = 1$ and $A_1 = 0$. Thus

$$\frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

Here is how we can use an expression like (2) to get an explicit formula for the a_k. For simplicity we only discuss 2nd order recurrences with $N = 2$ (the general case is similar). Suppose one can break apart the rational function into "partial fractions":

$$\frac{P(x)}{1-c_1x-c_2x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}. \tag{4}$$

Here A, B are constants, and α, β are the *inverses* of the roots of $1-c_1x-c_2x^2$. (This is slightly different from the partial fractions one usually sees in calculus, because we want the denominators to look like $1-\alpha x$, not $x-\alpha$ ...this reflects the fact that we use the inverses of the roots of the denominator of the left hand side.) Then one can conclude that

$$a_k = A\alpha^k + B\beta^k. \tag{5}$$

Why is (5) true? Because of the way the geometric series works:

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2x^2 + \alpha^3x^3 + \dots \tag{6}$$

Plug (6) into (4) along with the analogous series for $1/(1-\beta x)$, and then the cofficient of x^k is computed by the right hand side of (5).

For instance, consider the Fibonacci example. The roots of $1-x-x^2$ are $r_1 = (\sqrt{5}-1)/2$ and $r_2 = (-\sqrt{5}-1)/2$. The inverses of these are $1/r_2 = \alpha = (\sqrt{5}+1)/2$ and $1/r_1 = \beta = (-\sqrt{5}+1)/2$. So we get

$$a_k = A((\sqrt{5}+1)/2)^k + B((-\sqrt{5}+1)/2)^k. \tag{7}$$

We just have to solve for A and B. We can use the first two values of k for that:

$$1 = A + B$$
$$1 = A(\sqrt{5}+1)/2 + B(-\sqrt{5}+1)/2)$$

It's somewhat painful to solve for A and B, but if we do we find

$$A = (5+\sqrt{5})/10, \quad B = (5-\sqrt{5})/10.$$

Here's an application of an expression like (7). Since $|(-\sqrt{5}+1)/2| < 1$ and $(\sqrt{5}+1)/2) > 1$, it means that a_k must be very close to $A((\sqrt{5}+1)/2)^k$ for k large (because the other term in (7) will be close to 0). For example, when $k = 20$, the Fibonacci number is 10946, and

$$A((\sqrt{5}+1)/2)^{100} = 10945.999981728\dots$$

The relative error is 10^{-9}. If $k = 1000$ the relative error between the actual Fibonacci number and this approximation is 10^{-419}, an unimaginably small number. How small is unimaginably small? The *Planck length*, which supposedly is the distance at which quantum effects begin to dominate in spacetime, is about 10^{-35} m. The uncertainty principle of quantum mechanics means that we cannot measure distances smaller than this. Compared to our relative error, this number is gigantic.