MATH 455 EXAM I

This exam is worth 100 points, with each problem worth 20 points. There are problems on *both sides* of the page. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

When submitting your exam, please indicate which problems (including Problem 1) you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly five problems; any unselected problems will not be graded, and if you select more than five only the first five (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) Every graph has at least one spanning tree.
   (b) (4 pts) A walk in a graph is a trail with no repeated vertices.
   (c) (4 pts) A tree is a graph with no cycle.
   (d) (4 pts) The number of labelled trees of order n is n^{n-2}.
   (e) (4 pts) Every connected graph has at least one cut vertex

(2) Draw pictures of the trees with the following Prüfer codes:
   (a) (5 pts) $1, 2, 3, 4, 5, 6$
   (b) (5 pts) $1, 2, 3, 3, 2, 1$
   (c) (5 pts) $4, 4, 4, 4, 4, 4$
   (d) (5 pts) $1, 2, 1, 2, 1, 2$

(3) (20 pts) Let D be the graph in Figure 1. Compute the number of spanning trees for D.

(4) Let E be the graph in Figure 2.
   (a) (10 pts) Explain how we know that E is Eulerian.
   (b) (10 pts) Give two different Eulerian circuits for E (you can give your circuits by giving the sequence of vertex labels).

(5) Let F be the graph in Figure 3. Suppose the vertical edges have weight a, the diagonal edges have weight b, and the horizonal edges have weight c.
   (a) (7 pts) Find a minimal weight spanning tree if $0 < a < b < c$.
   (b) (7 pts) Find a minimal weight spanning tree if $0 < b < c < a$.
   (c) (6 pts) Is there any choice of values for a, b, c that would make F have a unique minimal weight spanning tree? Why or why not?

(6) Let $G = K_4$.
   (a) (6 pts) Write the adjacency matrix for G.
   (b) (6 pts) Let α_k be the number of k-step walks from any vertex to itself, and let β_k be the number of k step walks from any vertex to any other vertex. Compute $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$.
   (c) (8 pts) Show $|\alpha_k - \beta_k| = 1$ for any k. (Hint: First note the difference $\alpha_k - \beta_k$ is either ± 1 depending on the parity of k. Use induction.)

(7) Recall that a graph is called *d-regular* if every vertex has degree d.

Date: Thursday, 1 Mar 2018.

(a) (7 pts) Show that there is no 3-regular graph with an odd number of vertices.
(b) (6 pts) In contrast to (7a), there exists a 3-regular graph with $2n$ vertices for any $n \geq 2$. Draw 3-regular graphs with 4, 6, and 8 vertices.
(c) (7 pts) Show that for any d, there is no d-regular graph that has d vertices.
(d) (Extra credit 5 pts — not part of the 100 pts) Explain how to construct a 3-regular graph with $2n$ vertices for all $n \geq 2$.

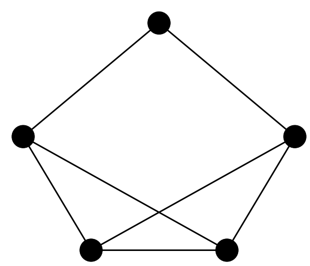

FIGURE 1. D

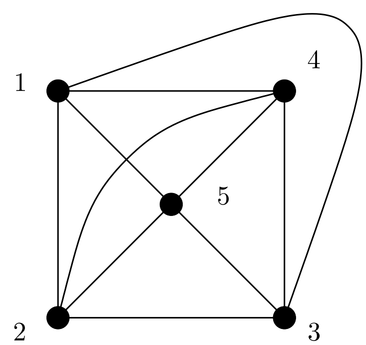

FIGURE 2. E

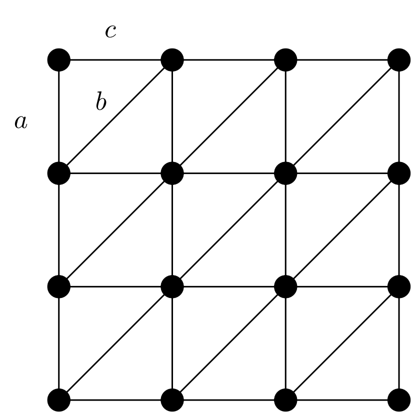

FIGURE 3. F