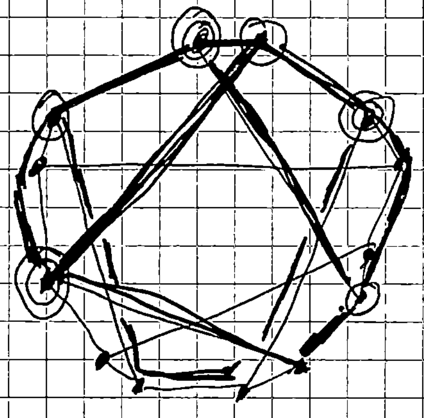
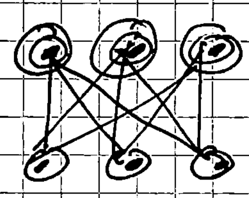
Answers to Exam II M455

1. (a) False. They can be negative.
   (b) True.
   (c) True. It is the multinomial coefficient $10!/4!6!0! = 210$
   (d) False. Kuratowski's theorem is about planarity.
   (e) False. Consider a graph with no edges or with one vertex and a loop.

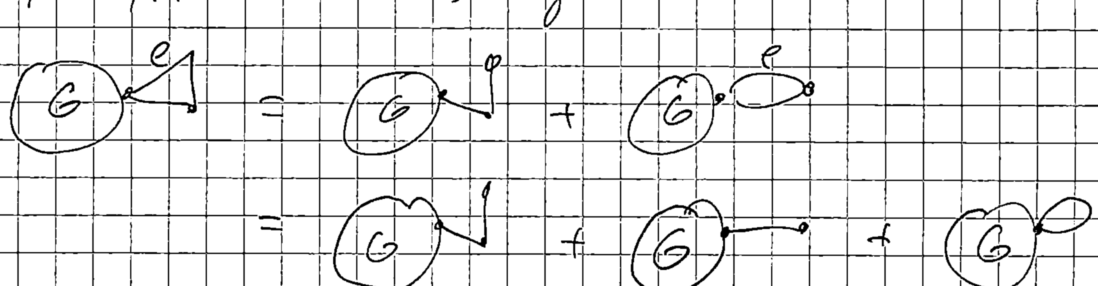
2. This graph is not planar. First contract an outside edge. Then the resulting minor graph contains a subdivision of $K_{3,3}$ and thus can't be planar by Kuratowski's theorem. (It is also possible to find the $K_{3,3}$ without contracting an edge, but it was easier for me to do it this way.)



an edge get contracted to get this vertex



3. (a) We apply the recursion $T_G(x,y) = T_{G-e} + T_{G/e}$ to the attached 3-cycle:



$= x^2 T_G + x T_G + y T_G$

$= (x^2 + x + y) T_G.$

(b) We peel off one triangle at a time. Each time we pick up a factor of x^2+x+y, as long as the triangle we remove is only attached at one vertex. We can do this: outside 6 first, then next 3, then final triangle. There are 10 triangles in total, so we get

$$(x^2+x+y)^{10}.$$

(4) (d) is the actual chromatic poly. Here's why the others can't be:

(a) : nonzero constant term
(b) : would be a graph with 3 verts and 4 edges
(c) : is negative at $k=1$
(e) : coeffs don't alternate in sign

(5) (a) $$k \cdot \frac{n!}{k!(n-k)!} \stackrel{?}{=} n \frac{(n-1)!}{(k-1)!\,(n-k)!}. \quad \text{Use}$$

$$\frac{k}{k!} = \frac{1}{(k-1)!}, \qquad n \cdot (n-1)! = n! \implies \text{identity true.}$$

(b) we know $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

now move the terms with k odd to the right.

(c) We know

$$(1+x)^n = \sum_{k=0}^{n} x^k \cdot \binom{n}{k}.$$

differentiate:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\, x^{k-1} \binom{n}{k}$$

set $x=1$:

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}.$$

You can also use (5a)

(6) ROCOCO $\Rightarrow$ RCCOOO

$\Rightarrow$ one R, 2 C's, 3 O's.

(a) multinomial coeff

$$\frac{6!}{1!\,2!\,3!} = \frac{6\cdot 5\cdot 4}{2!} = \boxed{60}$$

(b) We have to consider the different types, as we did in class.

			total
(1)	xxx	only 1: x = O.	1
(2)	xxy	2 choice for x (C, O) 2 choice for y after that 3 orderings	12
(3)	xyz	take {x, y, z} = {R, C, O} and order: 6 = 3!	6
			$\boxed{19}$

(c) Treat the three O's as a single character X. Then we have four things to order: R, C, C, X. We get the multinomial coeff

$$\frac{4!}{1!\,2!\,1!} = \boxed{12}$$

7 10 for city council, choose up to 4 — A
8 for school board, rank up to 3 — B
5 bond issues, approve or reject or say nothing — C
Partially completed ballots OK.
We compute A, B, C, and the total is the product A × B × C (because each section can be completed simultaneously and independently.)

A choose 0 1 2 3 4

$$\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4} = 386$$

B rank 0 1 2 3

$$1 + 8 + 8\cdot7 + 8\cdot7\cdot6 = 401$$

C there are 5 issues and each has 3 states (Y, N, say nothing) $\Rightarrow 3^5$ possibilities.
$= 243$

Total $= 386 \cdot 401 \cdot 243 = \boxed{37612998}$

8 8 secr. gen., pick 4 and rank — A
15 members = 10 non perm + 5 perm — B
name 10 in any order (so choose, not rank)
at least 2 must be perm.
Again the product A·B is the final answer, just like in 7

A $8\cdot7\cdot6\cdot5 = 1680$

B There are $\binom{15}{10}$ choices altogether but some are bad because they have < 2 permanent members. We have to take away the bad choices.
$\binom{10}{10}$ ways to pick with no perm mems.
$\binom{10}{9}\cdot\binom{5}{1}$ ways to pick with one perm mem.

$$\Rightarrow \binom{15}{10} - \binom{10}{10} - \binom{10}{9}\binom{5}{1} = 2943$$

Total $= \boxed{4944240}$