Math 411 Exam I Answers

(1) (a) True. Part of the definition
(b) False. Not every matrix is a power of a given matrix.
(c) False. The reciprocal of an integer isn't always an integer
(d) False. $(xy)^{-1} = y^{-1}x^{-1}$
(e) True. The binary operation on a subgp. comes from the big group.

(2) We have to check (a) associativity, (b) identity, and (c) existence of inverse.

(a) $(a*b)*c = (a+b-2)+c-2$
$= a+b+c-4$
$= a+(b+c-2)-2$
$= a*(b*c)$

(b) The identity is 2, because
$a*2 = a+2-2 = a = 2*a$
(c) The inverse is b s.t. $a*b=2$
This means $a+b-2=2$ or $b = 4-a$

(3) (a) according to the text, the orders of subgroups are the divisors of 40
This means 1, 2, 4, 5, 8, 10, 20, 40

(b) If $H \subset G$ has order d, it is generated by $g^{40/d}$. Therefore

Order of subgp	1	2	4	5	8	10	20	40
generator	g^{40}	g^{20}	g^{10}	g^{8}	g^{5}	g^{4}	g^{2}	g

(c) If $G = \langle g \rangle$ and $O(g) = 40$, the other generators are g^k with $GCD(k, 40) = 1$. This means

$$k = 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39.$$

(4) (a) yes, the group is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. in fact $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$.

(b) no, according to the book if G_1, G_2 are cyclic, then $G_1 \times G_2$ is cyclic iff $|G_1|$ is relatively prime to $|G_2|$. $GCD(2,4) = 2 \neq 1$.

(c) yes, same as in pt (b), except now $GCD(3,4) = 1$.

(5) we have $x * g = x$. Multiply both sides by x^{-1} on the left to get

$$\begin{aligned} x^{-1} * (x * g) &= x^{-1} * x \\ (x^{-1} * x) * g &= e \\ e * g &= e \\ g &= e. \end{aligned}$$

(c) We need to check (a) closed under multiplication (b) closed under taking inversion.

(a) $\begin{pmatrix} a & b \\ -2b & a \end{pmatrix}\begin{pmatrix} A & B \\ -2B & A \end{pmatrix} = \begin{pmatrix} aA - 2bB & aB + bA \\ -2(bA + aB) & -2bB + aA \end{pmatrix}$

and this is in G:

$(*)$ $\begin{pmatrix} \blacksquare = & \\ & \blacksquare \end{pmatrix}$ & $\begin{pmatrix} & \boxtimes \\ -2\boxtimes & \end{pmatrix}$

(b) The inverse of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is

$\frac{1}{\alpha\delta - \beta\gamma}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$. For $\begin{pmatrix} a & b \\ -2b & a \end{pmatrix}$ we get $\frac{1}{a^2 + 2b^2}\begin{pmatrix} a & -b \\ 2b & a \end{pmatrix}$

This is in G because (1) $a^2 + 2b^2 = 0 \Rightarrow$ both $a, b = 0$, which is not allowed, and (2) we still have the defining conditions $(*)$ true

(oops error in definition)

4

7 $Z(g) = \{ x \in G \mid gx = xg \}$

We need a) closed under multiplication and b) closed under taking inverses.

a) assume $x, y \in Z(g)$. Then

$$xyg = xgy = gxy$$

so $xy \in Z(g)$

b) assume $x \in Z(g)$. Then

$$xg = gx$$
$$xgx^{-1} = gxx^{-1} = g$$
$$(x^{-1}x)gx^{-1} = x^{-1}g$$
$$gx^{-1} = x^{-1}g$$

so $x^{-1} \in Z(g)$.

The definition I gave was for the center, not the centralizer. This is still a subgroup of G so I just graded that instead.