MATH 411 EXAM I

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem. G always denotes a group.
 (a) (4 pts) In any group $(G, *)$, the associative law holds: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
 (b) (4 pts) The group $GL_2(\mathbb{R})$ is an example of a cyclic group.
 (c) (4 pts) The nonzero integers form a group with the operation being ordinary multiplication.
 (d) (4 pts) If $x, y \in G$, then $(xy)^{-1} = x^{-1}y^{-1}$.
 (e) (4 pts) If G is abelian, then all subgroups $H \subset G$ are abelian as well.

(2) (20 pts) Let $\mathbb{Z}$ be given the binary operation $a * b = a + b - 2$. Show that $(\mathbb{Z}, *)$ is a group.

(3) Let G be a cyclic group of order 40 with a generator g.
 (a) (7 pts) What are the orders of the subgroups of G?
 (b) (7 pts) For each subgroup in part (3a), give a generator for it in terms of g.
 (c) (6 pts) Give a list of all generators of G in terms of g.

(4) Which of the following groups is cyclic and which aren't? Be sure to justify your answer.
 (a) (7 pts) The subgroup of $GL_2(\mathbb{R})$ of matrices of the forms $\left(\begin{smallmatrix} 1 & 2k \\ 0 & 1 \end{smallmatrix}\right)$, where $k \in \mathbb{Z}$.
 (b) (7 pts) The group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.
 (c) (6 pts) The group $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

(5) (20 pts) Let g be an element of a group $(G, *)$ and suppose that there exists an element $x \in G$ such that $x * g = x$. Show that g is the identity element of G.

(6) (20 pts) Let $G \subset GL_2(\mathbb{R})$ be the set of all matrices of the form $\left(\begin{smallmatrix} a & b \\ -2b & a \end{smallmatrix}\right)$, where $a, b \in \mathbb{R}$ and least one of a, b is nonzero. Show that G is a subgroup of $GL_2(\mathbb{R})$, where the group operation is matrix multiplication.

(7) (20 pts) Let G be a group and let $g \in G$ be a fixed element. Then the *centralizer* $Z(g)$ of g in G is the subset $Z(g) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Show that $Z(g)$ is a subgroup of G.

Date: Thursday, 1 March 2018.