MATH 797MF PROBLEM LIST

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Please complete 20 of these problems. You can hand them in at any time, but please try to submit them in groups of 5 at a time.

The problems cover a lot of different areas of the course; thus some are more geometric, some are more algebraic, etc. You can pick the problems to do that sound most appealling to you.

(1) (a) Prove that the action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ by fractional linear transformations is a left action.
   (b) Prove that the action is transitive, and that the stabilizer of i is isomorphic to $SO(2)$.

(2) (a) Prove that the left action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ preserves the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$ and the area $dx\, dy/y^2$.
   (b) Compute the area of the fundamental domain of $SL_2(\mathbb{Z})$ (with the hyperbolic measure).

(3) Let $\Gamma(N) \subset SL_2(\mathbb{Z})$ be the principal congruence subgroup of level N.
   (a) Show that $\Gamma(N)$ is torsion-free if $N > 2$. (Hint: use the fact that $\Gamma(N)$ is normal and that we know how to write down all torsion elements of $SL_2(\mathbb{Z})$.)
   (b) Show that the map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective (look at Shimura if you get stuck).

(4) The *Farey tessellation* is the tiling of $\mathfrak{H}$ by the $SL_2(\mathbb{Z})$-translates of the geodesic triangle with vertices at 0, 1, and ∞.
   (a) Show that if $N > 2$, a fundamental domain of $\Gamma(N)$ can be built from tiles in the Farey tessellation.
   (b) Draw pictures of $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$ for $N = 3, 4, 5, 6, 7$ with the triangulation induced from the Farey tessellation. (Hint: The vertices of this triangulation lie at the cusps. $3, 4, 5$ are going to look very familiar. For $6, 7$ you probably just want to draw a picture of a union of triangles with identifications on the boundary. It also helps to know that the cusps of $\Gamma(N)$ are in bijection with nonzero pairs $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ with $gcd(a, b, N) = 1$ modulo the relation $(a, b) \simeq (-a, -b)$.)

(5) To get presentation for a group using a fundamental domain, one can use the following theorem:

Date: Last modified: January 26, 2016.

Theorem 1. *Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete group acting properly discontinuously on $\mathfrak{H}$. Let $V \subset \mathfrak{H}$ be an open connected subset such that*

$$\mathfrak{H} = \bigcup_{\gamma\in\Gamma} \gamma V,$$
$$\Sigma = \{\gamma \mid V \cap \gamma V \neq \emptyset\} \text{ is finite.}$$

Then a presentation for Γ can by constructed by taking generators to be symbols $[\gamma]$ for $\gamma \in \Sigma$ subject to the relations $[\gamma][\gamma'] = [\gamma\gamma']$ if $V \cap \gamma V \cap \gamma' V \neq \emptyset$.

Use the theorem to get a presentation of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$. (Hint: take V to be a slight "thickening" of the fundamental doman D from class.)

(6) Let $\mathfrak{H}_3$ be hyperbolic three-space. An "upper halfspace" model for $\mathfrak{H}_3$ can by gotten by taking the points $(z, r) \in \mathbb{C} \times \mathbb{R}_{>0}$ and using the metric $ds^2 = (dx^2 + dy^2 + dr^2)/r^2$ (here we are writing $z = x + iy$). We can also think of $\mathfrak{H}^3$ as being the subset of quaternions $\mathbf{H} = \{x + iy + rj + tk \mid x, y, r, t \in \mathbb{R}\}$ with $r > 0$ and $t = 0$. Write $P = P(z, r)$ for the quaternion corresponding to $(z, r) \in \mathfrak{H}_3$.

Let $G = SL_2(\mathbb{C})$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, define a transformation of $\mathfrak{H}_3$ by

$$M \cdot P = (aP + b)(cP + d)^{-1}.$$

In this defintion the operations on the right are to be computed in $\mathbf{H}$.

(a) Show that this is a left action of G on $\mathfrak{H}_3$.
(b) Show that the action is transitive.
(c) Show that the stabilizer of $(0, j)$ is isomorphic to

$$SU(2) = \{M \in G \mid M\bar{M}^t = I\}.$$

(7) (a) Let $\Gamma = SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$. Then Γ acts on $\mathfrak{H}_3$. Show that the set

$$D = \{(x + iy, r) \in \mathfrak{H}_3 \mid 0 \leq |x|, y \leq 1/2, x^2 + y^2 + r^2 \geq 1\}$$

is a fundamental domain for the action of Γ on $\mathfrak{H}_3$. (Hint: generalize the algorithm from class that used S and $T^{\pm 1}$ to move points into the fundamental domain for $SL_2(\mathbb{Z})$).

(b) If you try to construct a fundamental domain of $\Gamma' = SL_2(\mathbb{Z}[\sqrt{-5}])$ using something like the above, it doesn't work. What goes wrong?

(8) (a) Show that if a point $\alpha \in \mathbb{R}$ is stabilized by a parabolic element in $SL_2(\mathbb{Z})$, then $\alpha \in \mathbb{Q}$.

(b) For any $\alpha \in \mathbb{R}$, denote its simple continued fraction expansion by $\alpha = [a_1, a_2, a_3, \dots]$ (look at any book on elementary number theory for a refresher of these concepts). Show that two points $\alpha, \alpha' \in \mathbb{R}$ are in the same $GL_2(\mathbb{Z})$ orbit if and only if their simple continued fractions have the same tail. (By definition, having the same tail means that if $\alpha = [a_1, a_2, \dots]$

and $\alpha' = [b_1, b_2, \dots]$, then there is some k, l such that $a_{k+i} = b_{l+i}$ for all $i \geq 0$.) Hint: consider the action of the matrix $\left(\begin{smallmatrix} a_1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(9) This is an easy exercise to help you see how various power series one encounters in number theory and combinatorics behave differently. Compute the coefficients of the product of two
   (a) power series: $\sum a_n x^n$
   (b) exponential series: $\sum a_n x^n/n!$
   (c) Dirichlet series: $\sum a_n/n^s$

(10) (a) Suppose a Dirichlet series $\sum_{n\geq 1} a(n)/n^s$ with $a(1) = 1$ can be written as an infinite product of the shape described in class:

$$\sum_{n\geq 1} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1},$$

where the product is taken over primes p and k is a fixed positive integer (This is called an *Euler product*). Prove
      (i) $a(mn) = a(m)a(n)$ if m and n are relatively prime.
      (ii) $a(p^n)$ can be computed in terms of $a(p^{n-1})$ and $a(p^{n-2})$. Compute the explicit formula for $a(p^n)$.
   (b) Verify that the Dirichlet series attached to the Eisenstein series $E_k(z)$, $k \geq 4$, has an Euler product. (Hint: relate the Dirichlet series to the Riemann zeta function somehow.)
   (c) Check the recursion for $a(p^n)$ for Δ for all p powers less than 100 (you probably want to use a computer for this).

(11) The E_8 root lattice Λ_8 can be described as the set of vectors in $\mathbb{R}^8$ with all components x_i either integral or half-integral (meaning odd integer/2) and such that $\sum x_i$ is an even integer. (Note that the x_i can't be a mixture of integers and half-integers …only one or the other). For instance $(1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2) \in \Lambda_8$, as is $(1, 1, 0, 0, 0, 0, 0, 0)$.
   (a) Verify that the q-expansion for the modular form built from the theta series for Λ_8 agrees with the Eisenstein series $E_4(z)$ up to as high a q-power as you dare. (q^n for $n \leq 3$ is probably possible without too much trouble by hand, but beyond this is probably going to require a computer.)
   (b) (Challenge—not part of the assignment) Is it possible to prove equality of this modular form with $E_4(z)$, without using $M_4(SL_2(\mathbb{Z})) = \mathbb{C}E_4(z)$?

(12) (a) Use a computer to find a basis of $M_k(SL_2(\mathbb{Z}))$ for $k \leq 36$. (Take each basis vector to be a q-series up to q^{30}.)

(b) Find an expression for the theta series of the Leech lattice[1] terms of your basis. (But don't look at the bottom of this encyclopedia entry, or you'll see spoilers.)

(c) Do the same for three Niemeier lattices of your choice (search for *Niemeier* at `oeis.org`; there are 24 of them).[2]

(d) Find the unique polynomials in E_4, E_6 giving the q-expansions you found in parts (b) and (c) (if that wasn't what you did for part (a)).

(13) Let $\theta(z)$ be the classical theta function

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

(a) Show that $\theta(z)^m = 1+\sum_{k\geq 1}\rho_m(k)q^k$, where $\rho_m(k)$ is the number of ways of representing k as a sum of m squares.

(b) One can show that $\theta(z)^4$ is a modular form of weight 2 for the group $\Gamma_0(4)$. Furthermore, one knows that the space $M_2(\Gamma_0(4))$ is spanned by the two weight two Eisenstein series $E_2(z)-2E_2(2z)$ and $E_2(z)-4E_2(4z)$, where $E_2(z) = 1-24\sum_{n\geq 1}\sigma_1(n)q^n$. (In particular these combinations of E_2 are actually modular.) Write $\theta(z)^4$ in terms of these Eisenstein series.

(c) Use part (b) to prove a famous formula of Jacobi:

$$\rho_4(n) = 8 \sum_{\substack{d\mid n \\ d\not\equiv 0 \bmod 4}} d.$$

(d) Deduce Lagrange's theorem: every positive integer can be written as a sum of four squares.

(14) The notation for this problem is taken from Problem (13). This time we consider $\theta(z)^8 \in M_4(\Gamma_0(4))$. This space of modular forms is spanned by $E_4(az)$ for $a = 1, 2, 4$. Prove

$$\rho_8(n) = 16 \sum_{\substack{d\mid n \\ d\not\equiv 2 \bmod 4}} d^3 + 12 \sum_{\substack{d\mid n \\ d\equiv 2 \bmod 4}} d^3.$$

(15) Let $f(z)$ be the eta-product $(\eta(z)\eta(11z))^2$.

(a) Compute the q-expansion of f up to q^{100}.

(b) Verify that the coefficients of this q-expansion agree with the $a(n)$ data produced by the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$ up to q^{100}. (For this you need to know that the factor for L-function of the elliptic curve at $p = 11$ is $(1-11^{-s})^{-1}$.)

[1] `http://oeis.org/A008408`

[2] These lattices don't appear to have their theta series programmed at `oeis.org`. You could submit your answer. **UPDATE 2016:** a Math 797MF alum did this already!

(16) The ring of *quasi-modular forms* on $SL_2(\mathbb{Z})$ is the polynomial ring $QM_* = \mathbb{C}[E_2, E_4, E_6]$ (in particular, there are no polynomial relations among these Eisenstein series). Define *Ramanujan's theta operator* Θ by

$$\Theta(f) = q\frac{df}{dq}.$$

f Thus if $f(z) = \sum_n a(n)q^n$, then $\Theta(f) = \sum na(n)q^n$. Show that Θ takes QM_* into itself. (Hint: to show two modular forms are equal you can check equality of q-expansions up to some degree. Going up to q^{50} is more than enough.)

(17) For $k > 2$ an even integer and for any nonnegative integer Δ, define

$$f_k(\Delta, z) = \sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=\Delta}} \frac{1}{(az^2+bz+c)^k}.$$

(We omit $a, b, c = 0$ if $\Delta = 0$). This sum converges absolutely.

(a) Show that f_k vanishes unless $\Delta \equiv 0, 1 \bmod 4$.
(b) Show that $f_k(\Delta, z)$ satisfies the transformation law of a modular form of weight $2k$ on $SL_2(\mathbb{Z})$. (In fact f_k is a modular form.)
(c) Show that $f_k(0, z)$ is a constant multiple of the Eisenstein series $E_{2k}(z)$.

(18) In this problem, we learn more about Dirichlet series built from elementary arithmetic functions. (If you don't know the definitions, try Wikipedia.) For each multiplicative function $f(n)$ below, express the associated Dirichlet series $\sum f(n)n^{-s}$ in terms of the Riemann zeta function. (Hint: these functions are all multiplicative, so you just have to match the factors for the primes p. Don't forget how Dirichlet series multiply (Problem 9).)

(a) $\mu(n)$, the Möbius function.
(b) $d(n)$, the number of positive divisors of n.
(c) $\varphi(n)$, Euler's phi-function.
(d) $\lambda(n) = (-1)^{\nu(n)}$, where $\nu(p_1^{r_1} \cdots p_k^{r_k}) = r_1 + \cdots + r_k$.
(e) $\mu(n)^2$.
(f) $d(n)^2$.
(g) $d(n^2)$.

(19) Let $G(n)$ be the number of finite abelian groups of order n, up to isomorphism. Build the associated Dirichlet series $L(s, G) = \sum_{n\geq 1} G(n)n^{-s}$.

(a) Prove that $G(n)$ is multiplicative.
(b) Find a formula for $G(p^r)$ in terms of partitions.
(c) Show (at least formally) that $L(s, G) = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\cdots$.
(d) Prove that $\sum_{n<X} G(n) \sim Cn$, where C is a constant. (Hint: use the Tauberian theorem mentioned in class.)
(e) Compute the constant C accurate to 5 places past the decimal.

(20) Show that the Hecke operators T_n satisfy

$$T_n T_m = \sum_{d|n,m} T_{mn/d^2}$$

when applied to any modular form for $SL_2(\mathbb{Z})$.

(21) Let $f \in S_k(SL_2(\mathbb{Z}))$ be a Hecke form. Then the Hecke eigenvalues generate a numberfield, called the *Hecke field* attached to f. It is conjectured that the degree of the Hecke field for f is the same as the dimension of S_k as a complex vector space. Check that this is so for the weights k such that $\dim S_k = 2$. (For example, in class we showed that the Hecke field for weight 24 is $\mathbb{Q}(\sqrt{144169})$.)

(22) Compute a basis of Hecke forms for $S_{36}(SL_2(\mathbb{Z}))$.

(23) Fix an integer q. Let $\mathcal{T}$ be the infinite tree of degree $q+1$. Thus $\mathcal{T}$ is a graph with infinitely many vertices and no cycles; each vertex of $\mathcal{T}$ is joined to $q+1$ others. Define the distance $d(v, v')$ between two vertices v, v' to be the length of the shortest path connecting them, where each edge is defined to have length 1. Finally define two sequences of correspondences $\theta_k, T_k, k \geq 0$ on the set of vertices of $\mathcal{T}$ by

$$\theta_k(v) = \sum_{d(v,v')=k} v'$$

and

$$T_k = \theta_k + T_{k-2} \quad (k \geq 2),$$

with the initial conditions $T_0 = \theta_0$, $T_1 = \theta_1$.

(a) Show that the θ_k satisfy

$$\theta_1\theta_1 = \theta_2 + (q+1)\theta_0,$$
$$\theta_1\theta_k = \theta_{k+1} + q\theta_{k-1} \quad (k \geq 2).$$

(b) Show that the T_k satisfy

$$T_k T_1 = T_{k+1} + qT_{k-1} \quad (k \geq 1).$$

(24) Let $p_k(t, n)$ be the polynomial in the Eichler–Selberg trace formula. Show that $p_k(t, n)$ is the coefficient of X^{k-2} in the expansion of $(1 - tX + nX^2)^{-1}$.

(25) The Eichler–Selberg trace formula for the trace $t_k(n)$ of T_n on $M_k(SL_2(\mathbb{Z}))$ is the same as the formula for $t_k^0(n)$, except that $-1/2\sum_{d|n}\min(d, n/d)^{k-1}$ is replaced with $+1/2\sum_{d|n}\max(d, n/d)^{k-1}$. Show that this is consistent with the decomposition $M_k = \mathbb{C}E_k \oplus S_k$.

(26) Since $M_2(SL_2(\mathbb{Z}))$ has dimension zero, the trace $t_2(n)$ of any Hecke operator T_n vanishes. Likewise since $S_4(SL_2(\mathbb{Z}))$ has dimension zero, the trace $t_4^0(n)$ vanishes.

(a) Use these facts to derive two recursive formulas for the Hurwitz-Kronecker class numbers $H(n)$.
(b) Use the formulas to make a table of $H(n)$ for $n \leq 48$.

(27) Let f be a weight k cusp form on $SL_2(\mathbb{Z})$. Show that the Mellin transform of $\varphi(t) = f(it)$ is $\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$.

(28) Let $f \in S_k^\varepsilon(N)$. Complete the details of the proof (using the "functional equation principle") to show that the completed L-function

$$\Lambda(s, f) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s, f)$$

satisfies the functional equation $\Lambda(s, f) = \varepsilon(-1)^{k/2}\Lambda(k - s, f)$.

(29) Let $f \in M_k(N)$ have Fourier expansion $\sum a_n q^n$. Show that $T_n f = \sum b_m q^m$, where

$$b_m = \begin{cases} a_0 \displaystyle\sum_{\substack{d|n \\ (d,N)=1}} d^{k-1} & \text{if } m = 0, \\ a_n & \text{if } m = 1, \\ \displaystyle\sum_{\substack{d|m,n \\ (d,N)=1}} d^{k-1} a_{mn/d^2} & \text{otherwise.} \end{cases}$$

(30) Fix a level N and consider the Hecke operators acting on weight k modular forms of level N. Write U_p for the operator T_p when $p|N$.
(a) Show that the operators satisfy
(i) $T_p T_{p^r} = T_{p^{r+1}} + p^{k-1}T_{p^{r-1}}$ if $(p, N) = 1$,
(ii) $T_{p^r} = (U_p)^r$ if $p|N$.
(iii) $T_m T_n = T_{mn}$ if $(m, n) = 1$.
(b) Conclude that if $f = \sum a_n q^n$ is a simultaneous eigenform for all T_p, U_p, and $a_1 = 1$, then $L(s, f)$ has the Euler product

$$L(s, f) = \prod_{(p,N)=1} (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \prod_{p|N} (1 - a_p p^{-s})^{-1}.$$

(31) (a) Suppose $N'|N$ and $M|(N/N')$. Suppose $f(z) \in S_k(N')$. Prove that $f(Mz) \in S_k(N)$.
(b) Suppose that f is a Hecke eigenform for all Hecke operators T_p with $(p, N) = 1$. Then prove each $f(Mz)$ is a Hecke eigenform with the same eigenvalues for all T_p with $(p, N) = 1$.

(32) This problem investigates the difficulties with Hecke operators and oldforms. Let $f \in S_2(11)$ be the newform $\eta(z)^2\eta(11z)^2$. Let $g_1 = f(z), g_2 = f(2z)$ be a basis for $S_2(22)$.
(a) Use problem 29 to write a formula for the q-expansion of T_p and U_p applied to the q-expansion of any weight k modular form on $\Gamma_0(N)$.

(b) Show that T_p ($p \neq 2, 11$) acts on $S_2(22)$ by the scalar matrix $\lambda_p Id$, where $\lambda_p = a_p(f)$. Thus any linear combination of g_1, g_2 is an eigenform for these operators.
(c) Show that U_2 can be diagonalized on $S_2(22)$ and compute the eigenforms g_1', g_2'.
(d) Show that g_1' and g_2' are not eigenforms for U_{11}. Thus the Hecke operators can't be diagonalized on $S_2(22)$.

(33) Show that $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ is in bijection with the cosets $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ via the "bottom row map:" $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c : d)$.

(34) Compute the space of modular symbols $\mathcal{M}_2(11)$ and the eigenvalues of the operators T_2, T_3, T_5. Check your answer by comparing with the newform from Problems 15 and 32, and by using the fact that $\mathcal{M}_2(N) \simeq S_2(N) \oplus \overline{S}_2(N) \oplus Eis_2(N)$.

(35) (a) Prove the recursion for the Chebyshev polynomials of the second kind $U_m(\cos\theta) = \sin(m+1)\theta / \sin\theta$: $U_m = 2xU_{m-1} - U_{m-2}$.
(b) Prove that the U_m form an orthogonal family of polynomials with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} fg\sqrt{1-x^2}\, dx$. In particular, prove

$$\langle U_m, U_n \rangle = 2\delta_{mn}/\pi.$$

(36) Compute the spectra of the following graphs. (One way to do it for the families is to find nice eigenvectors and prove that you have an eigenbasis.)
(a) The n-cycle.
(b) The n-star (one vertex joined to $n-1$ others; no other edges).
(c) The complete graph K_n.
(d) The complete bipartite graph $K_{n,n}$.
(e) The vertex-edge graphs of the octahedron and the icosahedron.

(37) Compute both sides of the graph trace formula for the triangle graph for the operators T_k, $k \leq 6$, and check that they agree.

(38) (a) Find the 6 admissible matrices in $PGL_2(\mathbb{F}_{13})$ that construct the Ramanujan graph $X^{5,13}$.
(b) Use the matrices to build a graph $Y^{5,13}$ with vertex set $\mathbb{P}^1(\mathbb{F}_{13})$, where z is joined to $\frac{az+b}{cz+d}$. Draw a picture.
(c) Compute the spectrum of $Y^{5,13}$ and verify that it is Ramanujan.