MATH 455 PROBLEM SET HINTS

PROBLEM SET 6

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§2.2.

(2) Use $n!/(n-1)! = n$ and the definition.
(3) Just use the definition of the binomial coefficients and start cancelling/rearranging.
(5) Use induction and the Addition property.
(6) Prove it the same way as (3). To see why "hexagon", look at where they live on the triangle.
(8) Use induction. $(x+y)^{\underline{n+1}} = (x+y)^{\underline{n}}(x+y-n)$. Then when you multiply the sum on the right by $(x+y-n)$, in the term indexed by k write it as $(x-k)+(y-n+k)$. Distribute and start collecting like terms, using the Addition identity for the binomial coefficients.

§2.3.

(1) Multiply both sides of (2.20) by $k_1!k_2!\cdots k_m!$. The LHS is $n!$ and the RHS is $(n-1)!\sum k_i$. Now use the fact that $\sum k_i = n$.
(2) Each path of a, b, c steps contributes to the monomial $x^a y^b z^c$ when you multiply out $(x+y+z)^{a+b+c}$ (in this correspondence a variable name corresponds to a step along that axis). So the total number of them is the coefficient of $x^a y^b z^c$ in this product, which is $\binom{a+b+c}{a,b,c}$.
(5) Set $x = y = 1$ in $(x+y+z)^n$ and then look at the coefficient of z^k.
(10) We first have to count orderings of the different 4-letter "selection patterns." The 4-letter patterns are $xxxx$, $xxxy$, $xxyy$, $xxyz$, $xyzw$ (this is the same as the number of partitions of 4). The ordering of any of these is the corresponding multinomial coefficient. So the number of orderings of $xxyz$, for example, is $\binom{4}{2,1,1}$ (notice that the partition shows up downstairs in the notation). These coefficients are

$$1, 4, 6, 12, 24.$$

Now we have to count fillings of any of these patterns. The letters we have at our disposal are $MB^2O^2P^2I^4S^4$ for Bobo and $MO^2P^2I^4S^6$ for Soso. The tricky thing when we fill is that we can't use every letter for every pattern entry, because we only have a limited number of some of them. (For instance, we can't use more than one M.) The number of ways to fill the patterns are (in the order of the patterns above)

$$\text{Bobo:} \quad 2,\ 2\cdot 5,\ \binom{5}{2},\ 5\cdot 5\cdot 4,\ 6\cdot 5\cdot 4\cdot 3$$

and

$$\text{Soso:} \quad 2,\ 2\cdot 5,\ \binom{4}{2},\ 4\cdot 4\cdot 3,\ 5\cdot 4\cdot 3\cdot 2.$$

When I multiply everything out and compute the difference I get 6408. (Let me know if you disagree!)

(11) It's the same as (10) but now we have to count orderings of the different 5-letter selection patterns. The 5-letter patterns are $xxxxx$, $xxxxy$, $xxxyy$, $xxxyz$, $xxyyz$, $xxyzw$, $xyzwu$ (again this is the same as the number of partitions of 5), and the multinomial coefficients are

$$1,\ 5,\ 10,\ 20,\ 30,\ 60,\ 120.$$

The sets of letters are $I^6U^5L^3W^2N^2K^2O^2A$ and $U^9K^3A^3H^2M^2N^2P$. I compute that there are 901182 for $LA\cdots$ and 349861 for $HU\cdots$