MATH 455 PROBLEM SET HINTS

PROBLEM SET 5

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.6.1.

(1) (a) 1 for K_1, 2 otherwise.
 (b) 2.
 (c) t (you need one color for each r_i).
 (d) 3.
 (e) 3.
 (f) tetra: 4. cube: 2. octa: 3. dodeca: 3. icosa: 4. These are found by giving explicit colorings.

(3) The chromatic number of the associated graph is 3. The groups are Moe, Mike, Jeff; Larry, John; and Curly has to be by himself.

(4) Suppose the chromatic number is k and the two verts are v and w. If there is a k-coloring where v and w get different colors, then the chromatic number doesn't increase when the edge between v and w is added. Otherwise, if every k-coloring gives v and w the same colors, then we need a new color when we join v and w. So the number goes up by at most 1.

§1.6.2.

(2) The largest degree of a vertex is 7 so $\omega(G) \leq 8$. It can't be 8 since it's not K_8. The other degrees aren't big enough to make it include K_7. It can't include K_6 for the same reason. So the biggest we can hope for is $\omega = 5$. And you can see K_5 in there, thus $\omega = 5$. To see that $\chi = 6$, notice that we need three colors on the right column and at least three on the left column (because those are odd cycles). You can check that 6 colors are enough.

(4) Suppose P is a detour path. If $\chi > \tau$, then there must be a vertex v that needs a different color from all the vertices in P. But this means it must be connected to all the vertices in P. So we can lengthen P by adding a new edge on the end, which contradicts P being a detour path.

§1.6.3.

(1) We know it's at most 4. We just need to make sure that it's not less than 4. Take a look at Kentucky (for instance). It borders 7 states, and they form an odd cycle around it. So we need 4 colors (because odd cycles need at 3 colors, and we need another for Kentucky).

(2) This time the chromatic number is 4 again. Look at Paraguay. It borders Brazil, Bolivia, and Argentina, and they border each other in pairs. So you need 4 colors. (I hope people printed out a map and tried to color it ... you'd find this configuration that way.) Note that there are only two countries in South America that don't border Brazil (a standard trivia question).

§1.6.4.

(1) (a) $t^4 - 3t^3 + 3t^2 - t$
   (b) $t^6 - 5t^5 + 10t^4 - 10t^3 + 5t^2 - t$
   (c) $t^4 - 4t^3 + 6t^2 - 3t$
   (d) $t^5 - 5t^4 + 10t^3 - 10t^2 + 4t$
   (e) $t^4 - 5t^3 + 8t^2 - 4t$
   (f) $t^5 - 9t^4 + 29t^3 - 39t^2 + 18t$

(2) At $k = 2$ we get -4. The values of chromatic polynomials at positive integers are always ≥ 0.

§Tutte.

(1) These are painful, but with patience they can be done. It helps to first prove a result of the following type. Let G be a graph, e an edge of it that is not a bridge, and write $G(n, e)$ for the (multi-) graph with n parallel copies of e. So $G(1, e)$ is the original graph, and $G(0, e)$ is the graph with e deleted. Let $G'(e)$ be the graph obtained by squashing e in $G = G(1, e)$. Then

$$T(G(n, e)) = T(G(0, e)) + (1 + y + y^2 + \cdots + y^{n-1})T(G'(e)). \tag{1}$$

This allows you to remove groups of parallel edges at the same time. (We did something like this in class, but we did the graph where 2 vertices are joined by n parallel edges.) Bunches of parallel edges are always created when you try to compute these polynomials.

   (a) Done in class: $x^{n-1} + x^{n-2} + \cdots + x + y$.
   (b) $x^3 + 2x^2 + (2y + 1)x + (y^2 + y)$. Note that it doesn't matter which edge you delete.
   (c) $x^5 + 3x^4 + (2y+4)x^3 + (5y+3)x^2 + (3y^2+4y+1)x + (y^3+2y^2+y)$.
   (d) $x^5 + 4x^4 + 10x^3 + (9y + 11)x^2 + (6y^2 + 15y + 5)x + (y^4 + 5y^3 + 9y^2 + 5y)$. Very painful.

(2) (a) Suppose the original edge in G is e and let e_1, e_2 be the two edges you get by subdividing e. Then G'/e_1 is the same as G, and $G' - e_1$ is $G - e$ with an extra bridge attached. So $T(G') = T(G) + xT(G - e)$.

   (b) Suppose G is an $(n-1)$-cycle, so that G' is an n-cycle. Then $G - e$ is a tree with $n-2$ edges and has Tutte polynomial x^{n-2}. So we get $T(C_n) = T(C_{n-1}) + x^{n-1}$. If we keep going we get the expected answer $x^{n-1} + x^{n-2} + \cdots + x + y$.

(3) Oops, there was a typo in the statement of the problem. $T(2,2)$ is actually the number of spanning subgraphs (not necessarily connected spanning subgraphs). The number of connected spanning subgraphs is $T(1,2)$. With this in mind, let's do the problem.

   The Tutte polynomial of K_4 is $T(x,y) = x^3 + 3x^2 + (4y+2)x + (y^3 + 3y^2 + 2y)$. We have $T(1,1) = 16 = 4^2$, which is what we expect. $T(1,2) = 38$. 16 of these are trees, so 22 are not. 12 are a triangle with an extra edge, 6 are two triangles joined along an edge, 3 are 4-cycles, and 1 is the graph itself. $T(2,2)$ is 64, so we need 26 more. 1 is the set of vertices, 6 are a single edge and the two other vertices, 3 are the pairs of nonadjacent edges, 12 are two edges meeting at a vertex + the other vertex, and 4 are a triangle and the other vertex.

(5) There are eight subsets of E. If you carefully compute all the quantities it works out.

§Reliability.

(1) Again these are very painful, especially K_5. It helps, like in the Tutte polynomial problems, to first see what happens when you delete bunches of parallel edges (I mean you should prove the analogue of equation (1) above for the reliability polynomial).
   (a) Substitute into the Tutte polynomial above and simplify to get $(1-p)^{n-1}(1+(n-1)p)$.
   (b) $24p^{10} - 60p^9 + 30p^8 + 20p^7 - 10p^6 - 5p^4 + 1$
   (c) $-31p^9 + 126p^8 - 180p^7 + 90p^6 + 9p^5 - 9p^4 - 6p^3 + 1$

(2) According to my computations, some of the roots do, and some don't . . .

§2.1.

(1) There are 52 letters, 1 underscore, and 10 digits. Words can't begin with a digit.
   (a) $53 \cdot 63^4$
   (b) $53 \cdot (1 + 64 + \cdots + 64^4)$ (the sum is taken over length 1, length 2, . . . , length 5)
   (c) $53 + 53 + 53 \cdot 63 + 53 \cdot 63 + 53 \cdot 63^2$ (because the last half of a word is determined by the first half, and again the sum is taken over increasing lengths)

(2) (a) Choose the spots for the vowels, then the vowels themselves, then the rest of the word: $\binom{11}{3} \cdot 5^3 \cdot 21^8$
   (b) The ones that have no repeated letters are $\binom{11}{3} \cdot 5 \cdot 4 \cdot 3 \cdot 21 \cdot 20 \cdot \cdots \cdot 14$. Now subtract this from (a).

(6) Break them up by 1st part only + 2nd part T/F + 2nd part Defs. These are disjoint possibilities, since to complete the exam we are only allowed to do one of them.
   (a) In the first part there are 2^4 possible answers for each question, and there are 10 questions. This gives $(2^4)^{10}$ ways to complete.
   (b) In the T/F part there are 8 questions and each has 2 possible answers. This gives 2^8 (it seems to me, the way the question is written, that one must answer them and can't leave them blank).
   (c) In the Defs part you must order 7 of ten things (because picking which definition goes with which term is the same as ordering them, and there are no repeats allowed; also I think we have to choose a definition for each term). This gives $10^{\underline{7}}$.

   The total is $(2^4)^{10} + 2^8 + 10^{\underline{7}}$.

(8) (a) Straights, including straight flushes: $10 \cdot 4^5$ (starting point times suit choices). Of these, 40 are straight flushes (10 starting points times 4 suits). So the difference is the answer.
   (b) Count all flushes and then subtract the straight flushes: $4 \cdot \binom{13}{5} - 40$.
   (c) 40
   (d) Pick which to take four of, then multiply by the choices remaining: $13 \cdot 48$.
   (e) Choose the two types of pair, then the number choices of each pair, then the number of cards remaining. There are $\binom{13}{2}$ types of pairs, then there are $6 = \binom{4}{2}$ ways to make a pair of a given type, then there are 44 cards left over to pick from (even though you've used 4 cards already and there are 48 left in the deck, there are 4 you can't pick). This gives $\binom{13}{2} \cdot 6^2 \cdot 44$.
   (f) Choose the type of pair, then 6 choices for the pair. Then choose the three remaining cards to be from distinct numbers with aribrary suit: $13 \cdot 6 \cdot \binom{12}{3} \cdot 4^3$.
   (g) One suit must appear twice. Choose it first, then choose the remaining 3 cards: $4 \cdot \binom{13}{2} \cdot 13^3$.
   (h) Similar to previous, but you don't have total freedom in picking the remaining three cards: $4 \cdot \binom{13}{2} \cdot 11 \cdot 10 \cdot 9$.
   (i) There are 4 suits possible for the three cards, and $\binom{13}{3}$ values that can be put there. After this there are 3 suits possible for the pair, and $\binom{13}{2}$ values that can go there. This gives altogether $4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2}$.

(11) Any divisor has the form $p_1^{a_1} \cdots p_m^{a_m}$ where $0 \le a_i \le n_i$. So there are $(n_1 + 1) \cdots (n_m + 1)$ total.