

MATH 455 PROBLEM SET HINTS

PROBLEM SET III

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.3.3.

- (2) Argue that if it has more than one, you can use the two spanning trees to make a cycle. For instance, let T be one spanning tree and T' be another. There is an edge in T' that's not in T . What happens if you add this edge to T ? (Compare with the next problem.)
- (4) If e is not in some spanning tree T , then adding e to T makes a cycle in G , and thus e couldn't have been a bridge.
- (5) In both cases the spanning tree constructed by the algorithm is unique. For the graph on the right it's clear, since all the weights are distinct (so there are no choices in the algorithm). For the graph on the left, there are choices made in the algorithm, but they all get made as the spanning tree is built.
- (6) Follow the description and see what edges are added in what order.

§1.3.4.

- (1) The labels of the leaves are never written down because leaves are always simply deleted during the encoding. Now let v be a vertex and suppose it isn't a leaf. If a vertex has degree d , then there are d vertices connected to it. If v survives to the end (it's one of the last two left), then $d-1$ of its neighbors must have been deleted and so it shows up $d-1$ times. On the other hand, if it doesn't survive to the end, then it will be deleted, but only after $d-1$ of its neighbors get deleted. So again it shows up $d-1$ times. (Note that this argument also applies to the leaves, since they have degree 1.)
- (2) (a) 221314415. (b) 5843339.
- (3) See Figure 1.
- (4) $K_{1,n}$, i.e. stars.
- (5) Path graphs.
- (6) This is a challenging problem. Here is one solution. Let A be the number of spanning trees of K_n containing a fixed edge. We need to compute $n^{n-2} - A$. Any spanning tree t determines a function f_t

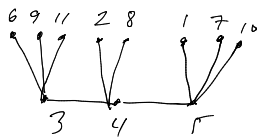


FIGURE 1.

on the edges E of K_n : if $e \in t$, then $f_t(e) = 1$, otherwise $f_t(e) = 0$. Let T be the set of spanning trees of K_n . Then we have

$$\sum_{e \in E} \sum_{t \in T} f_t(e) = A \cdot n(n-1)/2,$$

since there are $n(n-1)/2$ edges in E and since each edge appears in the same number of spanning trees. Now we apply a very powerful technique: we swap the order of summation.¹ This gives

$$\sum_{t \in T} \sum_{e \in E} f_t(e) = A \cdot n(n-1)/2.$$

The total sum hasn't changed, of course, but the point is now we can do the inner sum: any tree in T has $n-1$ edges, and when we sum f_t over the edges all we do is count the edges in t . Therefore

$$\sum_{t \in T} \sum_{e \in E} f_t(e) = \sum_{t \in T} (n-1) = A \cdot n(n-1)/2.$$

The sum in the middle is now simply $|T| \cdot (n-1) = n^{n-2} \cdot (n-1)$, and we can solve for A .

- (7) Do row/column operations to simplify the determinant computation.

¹This is the discrete analogue of changing the order of integration in Math 233.