

MATH 455 PROBLEM SET HINTS

PROBLEM SET II

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.3.1.

- (1) The best way to do this is to start from the 6 trees of order 6 and try adding edges. There are 11 trees of order 7.
- (3) Use induction. Call the two colors red and black. Clearly trees of order ≤ 3 are bipartite. Now take a tree T of order $n \geq 4$ and delete an edge such that the resulting two trees T', T'' have orders $< n$ and ≥ 2 . They are bipartite by assumption. If the vertices of the deleted edge have different colors, then we can restore the edge and T is bipartite. Otherwise, interchange the colors in T' and then rejoin to construct T .

§1.3.2.

- (1) By Theorem 1.11 a forest of order n with k components has $n - k$ edges. So to realize these examples, we need to be able to choose n and k as indicated to get the right number of edges. Only b, d, and e are possible.
- (2) Use the fact that there are an odd number of vertices and the sum of the degrees of the vertices is twice the number of edges.
- (5) If there is more than one path between u and v , then there must be a cycle.
- (7) Let u, v be the vertices of an edge e that is not a bridge. Then after deleting e there will still be a path from u to v . So there must be a cycle in the original graph.
- (10) Method 1: delete an edge e to break T into two smaller trees where the formula applies, and then consider the different possibilities when e is redrawn. Method 2: Call a vertex *internal* if it is not a leaf. The formula can be extended so that the sum is taken over all internal vertices (why?). Now we argue by induction over the number of internal vertices. The key point is that for some $d \geq 2$ we can find an internal vertex of degree d with $d - 1$ leaves. Deleting these leaves

reduces the number of internal vertices, so the formula holds. Now add the leaves back in, and what happens to the formula?