MATH 455 PROBLEM SET HINTS

PROBLEM SET I

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.1.1.

(1) Draw K_{10} with the vertices in a circle and then erase the diagonals that go directly across.
(2) Make a vertex for each person, and draw $A \to B$ is B is on A's list.
(3) Draw the graph with the indicated edges, and put a double edge if $*$ appears.

§1.1.2.

(1) The complete graph K_n has $n(n-1)/2$ edges (n choose 2).
(3) (a) Let N_k be the number of paths with k steps that start at c and walk away. Then we want $\sum_{k=0}^{4} N_k$. Think of counting sequences of distinct vertices. For instance, to get a walk of length 4, we need to write something like $cxyzw$, where x, y, z, w are some ordering of $\{a, b, d, e\}$. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such choices, so $N_4 = 24$. Similarly we find $N_0 = 1$, $N_1 = 4$, $N_2 = 12$, and $N_3 = 24$. So the total is 65. Note that this only counts the walks that have c at *one end* (which is how I interpreted it). If we want the walks that have c at either end, then the answer is $2 \cdot 65 + 1 = 131$ (why?). Either answer was accepted.
(b) This is very similar.
(c) 10.
(d) This is the same as the maximum length circuit in K_4. This is 4.
(9) The max will when the graph has two connected components, each of which are complete graphs. A little experimentation shows that we will want one component to be K_1 and the other to be K_{n-1}.
(11) If e is part of a cycle, it cannot be a bridge (for after you delete e you can still get make a walk from the endpoints of e to each other). Conversely, if e is not part of a cycle, it must be a bridge, since after deleting e there will be no path between its endpoints.

§1.1.3.

(2) Discussed in class.

(3) It is not, since $K_{4,4}$ contains no copy of K_3 as a subgraph, and K_3 is a subgraph of K_4 (cf. Thm 1.3)

(4) It is not, since it would have to contain all vertices of K_4. But the induced graph of the full set of vertices is the original graph K_4, which is not P_4.

§1.2.1.

(1) (Check me!) The radius is 3, the diameter is 6, and the center is the central vertex.

(2) I'll just do a few. To get these just do a bunch of examples and guess the connection between r, d and k. (r is the radius, d is the diameter.)

- (1) P_{2k}: $r = k$, $d = 2k - 1$.
- (2) C_{2k}: $r = d = k$.
- (3) K_n: $r = d = 1$.

(5) The key is the triangle inequality: if x is any vertex, then $d(x, u) \leq d(x, v) + 1$ and $d(x, v) \leq d(x, u) + 1$. So suppose $ecc(u) = a$ and $ecc(v) = a + k$, $k \geq 2$. Let x be a vertex with $d(x, v) = a + k$. Then $d(x, v) \leq d(x, u) + 1$, but $d(x, u)$ must be at most a, since $ecc(u) = a$. This gives $a + k \leq a + 1$, a contradiction.

§1.2.2.

(1) See the examples in class.

(2) You must count 3-step walks. We did a similar example in class.

(3) Any 2-step walk from a vertex v to itself is constructed by passing from v to a neighbor and back.