LIE GROUPS PROBLEMS

Please complete 20 of these problems. You can hand them in at any time.

The problems cover a lot of different areas of the course; thus some are more geometric, some are more algebraic, etc. You can pick the problems that sound most appealing to you.

Problems last modified: *Fri Mar 28 11:10:24 EDT 2014*.

(1) Hall, §1.9, 7.
(2) Hall, §1.9, 8.
(3) Hall, §1.9, 13.
(4) Hall, §2.10, 10.
(5) Hall, §2.10, 15.
(6) Hall, §2.10, 17.
(7) Hall, §2.10, 19.
(8) Hall, §3.9, 1.
(9) Hall, §3.9, 9.
(10) Hall, §4.11, 15.
(11) Hall, §4.11, 16.
(12) This problem has a companion, Problem (16). Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$, and for any i, j let E_{ij} be the elementary matrix with 1 in position i, j and 0 everywhere else.
   (a) Let $\mathfrak{h} \subset \mathfrak{g}$ be the subset of diagonal matrices. Show that $\mathfrak{h}$ is an abelian Lie subalgebra.
   (b) Assume $i < j$ and $k < l$, and compute $[E_{ij}, E_{kl}]$.
   (c) Fix $i < j$ and compute the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and E_{ij} (including how the brackets work).
   (d) Fix $i < j$ and $k < l$ and compute the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and E_{ij}, E_{kl}.
(13) (a) Let $\mathrm{O}_{2n}(\mathbb{C})$ be defined using the symmetric bilinear form $\langle x, y\rangle = \sum_{i=1}^{2n} x_i y_{2n+1-i}$, and suppose $X = (X_{ij}) \in \mathfrak{gl}_{2n}(\mathbb{C})$. Compute the conditions on the X_{ij} that imply $X \in \mathfrak{o}_{2n}(\mathbb{C})$.
   (b) Do the same for $\mathrm{O}_{2n+1}(\mathbb{C})$ using the form $\langle x, y\rangle = x_{2n+1}y_{2n+1} + \sum_{i=1}^{2n} x_i y_{2n+1-i}$.
(14) Let $\mathrm{Sp}_{2n}(\mathbb{C})$ be defined using the antisymmetric bilinear form $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_{2n+1-i} - x_{2n+1-i} y_i$, and suppose $X = (X_{ij}) \in \mathfrak{gl}_{2n}(\mathbb{C})$. Compute the conditions on the X_{ij} that imply $X \in \mathfrak{sp}_{2n}(\mathbb{C})$. (If you want to use a different nondegenerate antisymmetric bilinear form that's OK. One advantage to this one is that the answer exhibits some similarity with the even orthogonal case (Problem (13b)).)
(15) Using Problems (13)–(14), compute the dimensions (over $\mathbb{C}$) of $\mathfrak{o}_{2n}(\mathbb{C})$, $\mathfrak{o}_{2n+1}(\mathbb{C})$, and $\mathfrak{sp}_{2n}(\mathbb{C})$.

(16) In this problem you will study the structure of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. This is mainly a matter of properly interpreting the computations you did in Problem (12). We will use the previous notation.
  (a) Show that each E_{ij}, $i \neq j$, spans a root space for $\mathfrak{h}$.
  (b) Let $\varepsilon_k \colon \mathfrak{h} \to \mathbb{C}$ be the linear form such that $\varepsilon_k(\mathrm{Diag}(h_1, \dots, h_n)) = h_k$. Compute the root for the root space $\mathbb{C}E_{ij}$ in terms of these linear forms.
  (c) Let Φ be the set of roots. What is the cardinality of Φ? Verify that we have a decomposition
$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$
  (d) Verify that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is (i) $\mathfrak{g}_{\alpha+\beta}$, (ii) 0, or (iii) a subspace of $\mathfrak{h}$, depending on whether (i) $\alpha+\beta \in \Phi$, (ii) $\alpha+\beta \notin \Phi$ and $\beta \neq -\alpha$, or (iii) $\alpha + \beta = 0$.

(17) In this problem we investigate the geometry of the roots for $\mathfrak{sl}_n(\mathbb{C})$. The set of roots Φ is called the *root system of type* A_{n-1}.
  (a) Using the forms ε_i from the previous problem, we have represented Φ as a subset of $\mathbb{C}^n$. Explain why Φ is actually a subset of the *real* subspace $V = \mathbb{R}\varepsilon_1 \oplus \cdots \oplus \mathbb{R}\varepsilon_n \subset \mathbb{C}^n$.
  (b) In fact more is true: the real linear subspace spanned by Φ in $\mathbb{C}^n$ is a hyperplane $W \subset V$. Why? This means we can draw the root system for $\mathfrak{sl}_n(\mathbb{C})$ in $\mathbb{R}^{n-1}$. (This is also why the subscript for the A is $n-1$, not n.) Show how A_1 sits inside $\mathbb{R}^2$, and A_2 sits inside $\mathbb{R}^3$.
  (c) The standard euclidean dot product on $\mathbb{R}^n$ allows us to talk about the lengths and angles between vectors in A_{n-1}. Verify that with this inner product the 6 roots of A_2 form the vertices of a regular hexagon.
  (d) Draw a picture of the root system A_3 in $\mathbb{R}^3$ with the correct angles between root vectors. (Hint: there are several sub root systems of type A_2 that correspond to different copies of $\mathfrak{sl}_3(\mathbb{C})$ inside $\mathfrak{sl}_4(\mathbb{C})$. These are regular hexagons. Every root is contained in 2 of these hexagons. It helps to think in terms of these hexagons. Also some pairs of roots are perpendicular.)

(18) (a) Prove that $\mathfrak{b}$ (upper triangular matrices in $\mathfrak{gl}_n(\mathbb{C})$) is a solvable Lie algebra.
  (b) Prove that $\mathfrak{u}$ (strictly upper triangular matrices in $\mathfrak{gl}_n(\mathbb{C})$) is a nilpotent Lie algebra.

(19) (a) Compute the Killing form on $\mathfrak{sl}_n(\mathbb{C})$. Verify that $\mathfrak{sl}_n(\mathbb{C})$ is semisimple.
  (b) Compute the Killing form on $\mathfrak{gl}_n(\mathbb{C})$. Verify that $\mathfrak{gl}_n(\mathbb{C})$ is not semisimple.
  (c) Compute the Killing form on $\mathfrak{b}$ (see problem (18a)). Is $\mathfrak{b}$ semisimple?

(20) Let $\alpha, \beta \in \Phi$, where $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Check that if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then either $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$ or $[X, Y] = 0$.

(21) Prove that if $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to the Killing form.

(22) Let $\alpha, \beta \in V$ be roots. Recall that $2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle$ is an integer, where $\langle\ ,\ \rangle$ is the inner product on V induced from the Killing form. Classify what the possibilities are for the angle θ between α, β and for the ratio of the length of β to that of α. (Hint: first show that $4\cos^2(\theta)$ is an integer.)

(23) Let $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices (we use the "off-diagonal" version of the symplectic pairing). Let $e_i \in \mathfrak{h}^*$ $(i = 1, 2)$ be the linear form that takes $\mathrm{Diag}(a_1, a_2, -a_2, -a_1)$ to a_i.
   (a) Show that the roots of $\mathfrak{g}$ are $\pm 2e_1$, $\pm 2e_2$, $\pm(e_1 + e_2)$, $\pm(e_1 - e_2)$ by finding the root spaces in $\mathfrak{g}$. (Hint: try elementary matrices or matrices that are very close to elementary.)
   (b) Generalize this to $\mathfrak{sp}_{2n}(\mathbb{C})$.

(24) Suppose $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ is a Cartan subalgebra, Φ is the set of roots, and B is the Killing form. Show that for $H, H' \in \mathfrak{h}$, we have

$$B(H, H') = \sum_{\alpha\in\Phi} \alpha(H)\alpha(H').$$

(25) Let V be the real span of the roots Φ in $\mathfrak{h}^*$ and $\langle\ ,\ \rangle$ the inner product on V induced by the Killing form. For any $\alpha \in \Phi$, let $\alpha^\vee$ be the corresponding coroot. Prove that the linear map

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha$$

is an orthogonal reflection on V. That is, s_α takes α to $-\alpha$, fixes the hyperplane perpendicular to α, and is orthogonal with respect to $\langle\ ,\ \rangle$.

(26) Draw convincing pictures of all the irreducible root systems of ranks ≤ 4.

(27) This exercise realizes the exceptional Lie algebra $\mathfrak{g}_2(\mathbb{C})$ as a subalgebra of $\mathfrak{so}_8(\mathbb{C})$. A computer is probably going to be helpful for this, unless you *really* like multiplying big matrices together. Let E_{ij} be the elementary matrix with a 1 in row i and column j and 0s everywhere else. For a matrix A let A° be the transpose of A about the *opposite* diagonal. For example,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^\circ = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Put $A^* = A - A^\circ$.
   (a) Draw a picture of the root system for $\mathfrak{g}_2$ with the simple roots labelled. (Make α_2 the long simple root.)
   (b) Write the positive roots as linear combinations of the simple roots.

(c) (All the matrices in what follows are 8×8.) Let $X_{\alpha_1} = E^*_{12} + E^*_{34} + E^*_{35}$, $X_{\alpha_2} = E^*_{23}$, $H_{\alpha_1} = \mathrm{Diag}(1, -1, 2, 0, 0, 0, 0, 0)^*$, $H_{\alpha_2} = \mathrm{Diag}(0, 1, -1, 0, 0, 0, 0, 0)^*$. Show that the X_{α_i} generate the root spaces $\mathfrak{g}_{\alpha_i}$ by computing $\mathrm{ad}_{H_{\alpha_1}}$, $\mathrm{ad}_{H_{\alpha_2}}$ on them (ooh, triple subscript!).

(d) Compute the appropriate brackets of X_{α_1}, X_{α_2} as indicated by the root system, and show that the H_{α_i} act on these vectors as expected.

(28) (a) Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let V be the standard representation of $\mathfrak{g}$. Compute the multiplicity of the trivial representation in $V^{\otimes m} := V \otimes \cdots \otimes V$ (m factors) for $m \leq 10$.

(b) Make a conjecture about these multiplicities and prove your conjecture. (They are related to a famous sequence in mathematics.)

(c) Now consider the same computation but with $\mathfrak{g}$ replaced by one of the exceptional algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, and $\mathfrak{e}_8$, and with V replaced by the appropriate adjoint representation. Compute a table of multiplicities of the trivial representation for small values of m. (You are absolutely going to need to use a computer for this; I recommend `LiE` or `sage`. Or look for tables online.) Do you notice anything? (Deligne did.)

(29) Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$. Let V be $\mathbb{C}^{n+1}$ with the standard action of $\mathfrak{g}$: $X \cdot v = Xv$ (matrix multiplication). Let $V_k = \bigwedge^k(V)$.

(a) Explain how $\mathfrak{g}$ acts on elements of V_k.

(b) Show that V_k is an irreducible representation of highest weight ϖ_k, where $\{\varpi_1, \ldots, \varpi_n\}$ is the set of fundamental weights. (Hint: compute all the weights in V_k and show that a unique one is dominant and is ϖ_k. Then argue that this implies the representation is irreducible.)