Classification of simple Lie groups

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1. Simple Lie groups were first listed by Killing in 1890. The first complete proof of Killing's results was given by É. Cartan (1894). Basing on the paper by H. Weyl [2], van der Waerden proposed in 1933 [1] a new, more geometric, method of the classification of simple Lie groups. In this note, we prove that a semisimple Lie group is determined by its system of simple roots and reduce the problem of listing all simple Lie groups to the following simple geometric problem: to construct all bases of a n-dimensional Euclidean vector space such that $\frac{2(a,b)}{(b,b)}$ is a non-positive integer for any two distinct vectors a and b (here (a,b) is the scalar product of a and b).

2. H. Weyl assigns to any semisimple Lie group $\mathfrak{G}$ with complex parameters the system $\Sigma(\mathfrak{G})$ of its roots that determines $\mathfrak{G}$ completely. Here $\Sigma = \Sigma(\mathfrak{G})$ is a finite subset of a n-dimensional real Euclidean vector space R^n satisfying the following conditions:

2 (1) If $a \in \Sigma$, then $-a \in \Sigma$, but $ka \notin \Sigma$ for $k = 2, 3, \ldots$

2 (2) Let a and b be two different roots. If $b + ia \in \Sigma$ for $-p \leq i \leq q$, while $b - (p+1)a \notin \Sigma$ and $b + (q+1)a \notin \Sigma$, then $p - q = \frac{2(b,a)}{(a,a)}$.

2 (3) If two systems $\Sigma(\mathfrak{G}_1)$ and $\Sigma(\mathfrak{G}_2)$ are similar, i.e., are transformed one into another by a homothety of R^n, then they coincide.

If, in particular, $\mathfrak{G}$ is a simple group, then

2 (4) $\Sigma(\mathfrak{G})$ cannot be split into two orthogonal subsets Σ_1 and Σ_2.

3. Let us give examples of simple groups and write down their root systems. These examples were studied in detail by H. Weyl [2].

– Originally published in Matematicheski Sbornik, vol. 18(60) (1946), 347–352. English translation by A.L. Onishchik (1997). A detailed presentation of the structure and classification theory of complex semisimple Lie algebras based on the method of simple roots is given in E.B. Dynkin, Structure of semisimple Lie algebras, Uspekhi Mat. Nauk, vol. 2 (1947), 59–127; English transl., Amer. Math. Soc. Transl. (1), vol. 9, Amer. Math. Soc., Providence, RI, 1962, 328–469.

A_n — the group of linear transformations with determinant 1 of a $(n+1)$-dimensional complex vector space L^{n+1}.

$\Sigma(A_n) = \{e_p - e_q\}_{p,q=1}^{n+1}$ ($p \neq q$; $e_1, \dots, e_{n+1}$ is an orthonormal basis of R^{n+1}).

B_n — the group of orthogonal transformations of L^{2n+1}.

$\Sigma(B_n) = \{\pm e_p, \ \pm e_p \pm e_q\}_{p,q=1}^{n}$ ($p \neq q$).

C_n — the group of symplectic transformations of L^{2n}, i.e. of linear transformations leaving invariant the differential form

$$\sum_{k=1}^{n} (x_k dx_{n+k} - x_{n+k} dx_k).$$

$\Sigma(C_n) = \{\pm 2e_p, \ \pm e_p \pm e_q\}_{p,q=1}^{n}$ ($p \neq q$).

D_n — the group of orthogonal transformations of L^{2n}.

$\Sigma(D_n) = \{\pm e_p \pm e_q\}_{p,q=1}^{n}$ ($p \neq q$).

4. A vector of R^n will be called *positive* if its first non-zero coordinate is positive. The subset P of all positive vectors satisfies the following conditions:

4 (1) Suppose $a \neq 0$. Then either $a \in P$ or $-a \in P$, but it is impossible that $a \in P$ and $-a \in P$.

4 (2) If $a \in P$, $b \in P$, $\lambda > 0$, $\mu \geq 0$, then $\lambda a + \mu b \in P$.

We will write $a > 0$ whenever $a \in P$, and $a < 0$ whenever $-a \in P$.

LEMMA I. *If vectors $a_1, \dots, a_p$ are positive and $(a_i, a_k) \leq 0$ ($i, k = 1, \dots, p$; $i \neq k$), then these vectors are linearly independent.*

Indeed, suppose that $a_p = \sum_{i=1}^{p-1} \lambda_i a_i = \sum' \lambda_i a_i + \sum'' \lambda_i a_i$, where $\sum'$ contains the summands with positive coefficients λ_i, while $\sum''$ contains the summands with negative ones. Set $b = \sum' \lambda_i a_i$, $c = \sum'' \lambda_i a_i$. Then $(b, c) \geq 0$, $a_p = b + c$, where $c \leq 0$ and hence $b \neq 0$. We have $(a_p, b) = (b, b) + (c, b) > 0$, but, on the other hand, $(a_p, b) = \sum' \lambda_i (a_p, a_i) \leq 0$.

5. A positive root a is called *simple* if it cannot be decomposed into the sum of two positive roots. Any positive root can be expressed as the sum of simple roots.

If b is a positive root and a is a simple root, then $a - b$ cannot be a positive root. Hence, the difference of two simple roots a_1 and a_2 is not a root, and by 2 (2) we get $\frac{2(a_1, a_2)}{(a_1, a_1)} = -q \leq 0$. Thus $(a_1, a_2) \leq 0$, and, by Lemma I, the simple roots are linearly independent. Any positive root decomposes uniquely into the sum of simple ones.

We say that a positive root has *order* k, if it is the sum of k simple roots. Let us show that any root c of order k has the form $a + b$, where a is a simple root and b a root of order $k - 1$. Indeed, if $a_1, \dots, a_n$ is the system of all simple roots, then $c, a_1, \dots, a_n$ are linearly dependent, and, by Lemma I, at *least one scalar product* (c, a_i) *is positive.* This means that in 2 (2) we have $p \neq 0$, and hence $c - a_i$ is a root.

6.

THEOREM I. *A semisimple group* $\mathfrak{G}$ *is completely determined by the system* $\Pi(\mathfrak{G})$ *of its simple roots.*

To prove this, it is sufficient to construct all roots of the group $\mathfrak{G}$ starting from its simple roots. By 2 (1), we can restrict ourselves by constructing all positive roots. All roots of order 1 are given, since these are the simple roots. Suppose that we have already constructed all roots of any order $< k$. Any root of order k has the form $b + a$, where b is a root of order $k - 1$ and a is a simple root (see no. 5). The formula $q = p - \frac{2(b,a)}{(a,a)}$ (see 2 (2)) allows to decide whether the sum of a simple root and a root of order $k - 1$ is a root. Indeed, all roots in the sequence $b, b - a, b - 2a, \ldots$ are positive of order $< k$, and thus p is known by the inductive hypothesis. Therefore we can construct all the roots of order k.

7. It is not difficult to determine the systems of simple roots for the groups given in no. 3:

$$\begin{aligned} \Pi(A_n) &= \{e_p - e_{p+1}\}_1^n; \\ \Pi(B_n) &= \{e_p - e_{p+1}, e_n\}_1^{n-1}; \\ \Pi(C_n) &= \{e_p - e_{p+1}, 2e_n\}_1^{n-1}; \\ \Pi(D_n) &= \{e_p - e_{p+1}, e_{n-1} + e_n\}_1^{n-1}. \end{aligned}$$

8. A finite subset Γ of R^n will be called a (Π)-*system* if it satisfies the following conditions:

8 (1) If $a \in \Gamma$ and $b \in \Gamma$, $a \neq b$, then $\frac{2(a,b)}{(a,a)}$ is a non-positive integer.
8 (2) Γ is linearly independent.
8 (3) Γ cannot be split into two orthogonal subsets.

By 2 (2), 2 (4) and no. 5, we have:

THEOREM II. *The system* $\Pi(\mathfrak{G})$ *of simple roots of a simple Lie group* $\mathfrak{G}$ *is a* (Π)*-system.*

Theorems I and II reduce the problem of classification of simple Lie groups to that of construction of all possible (Π)-systems.

9. Let a and b be two distinct vectors of a (Π)-system Γ. Then the angle $\widehat{(a,b)}$ between a and b is equal either 90° or 120° or 135° or 150°.

Indeed, since $\frac{2(a,b)}{(a,a)}$ and $\frac{2(a,b)}{(b,b)}$ are integers, we have

$$4\cos^2 \widehat{(a,b)} = \frac{2(a,b)}{(a,a)} \frac{2(a,b)}{(b,b)}$$

is also an integer and therefore it is equal to 0,1,2 or 3. Thus, the only possible values of $\cos \widehat{(a,b)}$ are 0, $-\frac{1}{2}$, $-\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{3}}{2}$.

10. Assign to any element of a given (Π)-system Γ a vertex of a diagram. Connect two vertices by a single, double or triple edge if the corresponding angle is equal to 120°, 135° or 150° respectively. Do not connect any pair of vertices corresponding to orthogonal vectors. The diagram constructed in this way will be called the *angle diagram* of Γ. By writing under every vertex the squared length (a, a) of the corresponding vector a, we get a diagram that determines Γ completely. It is called the *diagram* of the system Γ.

As an example, we draw the diagrams of the systems $\Pi(A_n)$, $\Pi(B_n)$, $\Pi(C_n)$, $\Pi(D_n)$.

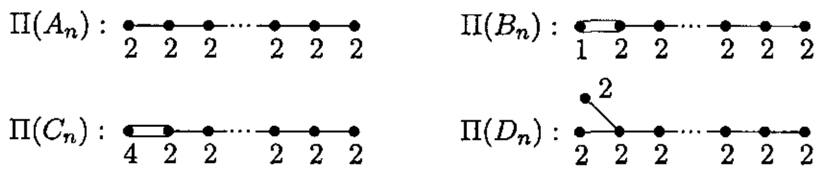

FIGURE 1

11.

LEMMA II. *The angle diagram of a* (Π)*-system cannot be one of the diagrams* $I_1 - I_2$, $II_1 - II_4$, $III_1 - III_5$ *(see Figure 2).*

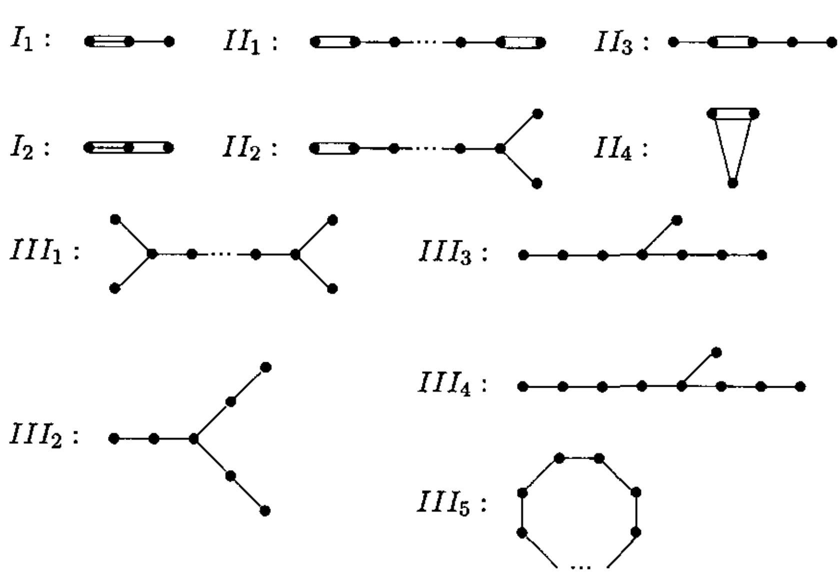

FIGURE 2

Suppose that a (Π)-system Γ has one of the listed diagrams as its angle diagram. Let $a_1, \dots, a_p$ be the vectors of Γ. Set $b_i = \lambda_i a_i$, where $\lambda_i \neq 0$ ($i = 1, \dots, p$). Then

$$\sum_{i=1}^{p}\sum_{k=1}^{p}(b_i, b_k) = \left(\sum_{i=1}^{p} b_i\right)\left(\sum_{i=1}^{p} b_i\right) > 0.$$

We arrive at a contradiction by choosing the lengths of b_i in such a way that

$$\sum_{i=1}^{p}\sum_{k=1}^{p}(b_i, b_k) \leq 0.$$

The diagrams $I'_1 - I'_2$, $II'_1 - II'_4$, $III'_1 - III'_5$ (see Figure 3) show, how to do this. The values (b_i, b_i) are indicated there under the corresponding vertices, while (b_i, b_k) are written over the corresponding edges.

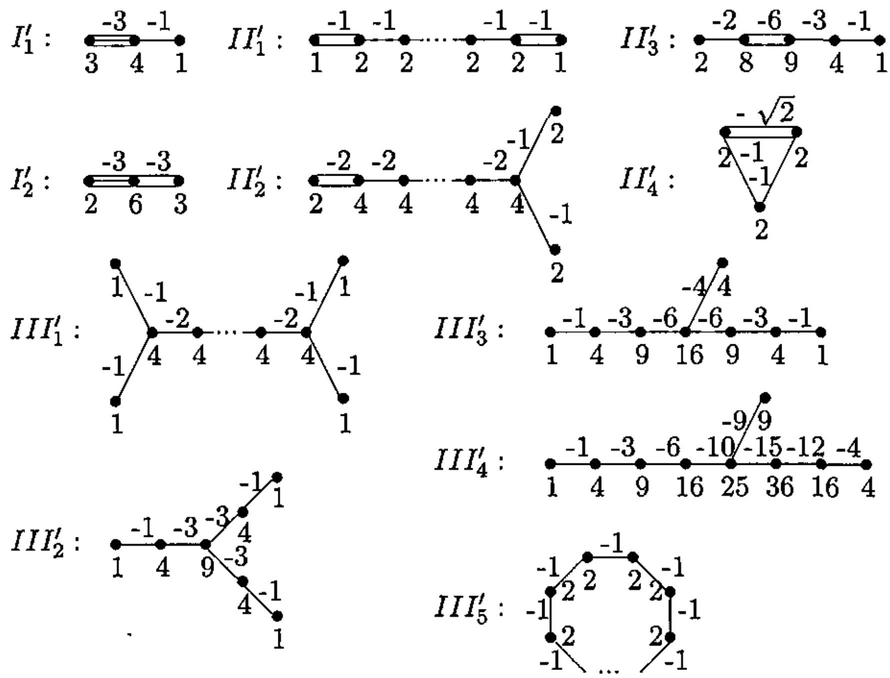

FIGURE 3

Clearly, Lemma II can be strengthened in the following way: *the angle diagram of a* (Π)*-system does not contain any subdiagram of the form* $I_1 - III_5$.

12.

LEMMA III. *The angle diagram of an arbitrary* (Π)*-system is one of the diagrams* I, $II^1 - II^2$, $III^1 - III^5$ *(see Figure 4).*

Indeed, a diagram that contains a triple edge and is different from I necessarily contains, as a subdiagram, one of the diagrams $I_1 - I_2$ of Lemma II which is impossible. Similarly, if a diagram contains a double edge, then, by $II_1 - II_4$, it coincides with one of the diagrams $II^1 - II^2$. Finally, for a diagram without triple or double edges, one excludes any possibility except of $III^1 - III^5$ with the help of $III_1 - III_5$.

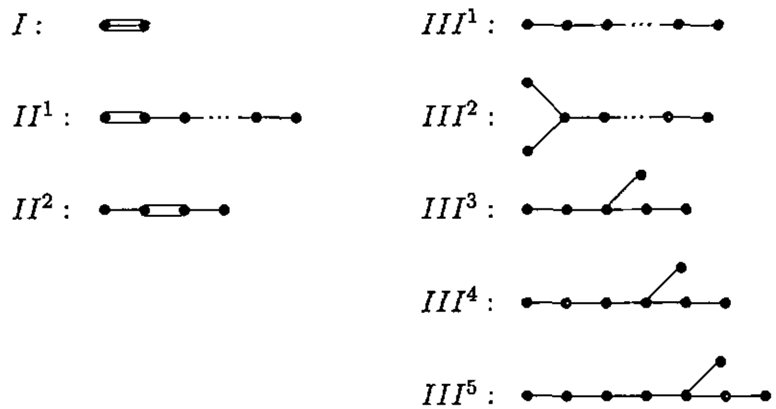

FIGURE 4

13. Let a and b be two vectors of a (Π)-system Γ such that $\widehat{(a,b)} = 120°$. Then

$$\frac{2(a,b)}{(a,a)}\frac{2(a,b)}{(b,b)} = 4\cos^2\widehat{(a,b)} = 1.$$

By 8 (1),

$$\frac{2(a,b)}{(a,a)} = \frac{2(a,b)}{(b,b)} = -1.$$

Thus $(a,a) = (b,b)$. In the same way we get $(a,a) = 2(b,b)$ whenever $\widehat{(a,b)} = 135°$, and $(a,a) = 3(b,b)$ whenever $\widehat{(a,b)} = 150°$ (supposing that $(a,a) \geq (b,b)$). This remark and Lemma III imply the following theorem.

THEOREM III. *An arbitrary* Π*-system either is similar to one of the systems* $\Pi(A_n)$, $\Pi(B_n)$, $\Pi(C_n)$, $\Pi(D_n)$ *or is given by one of the diagrams drawn on Figure 5 (the proportionality factor* λ *being an arbitrary positive number).*

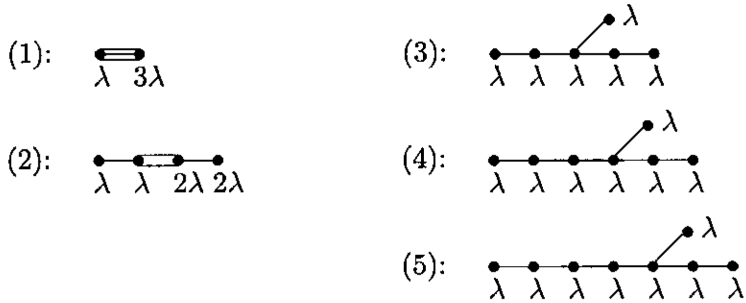

FIGURE 5

14. Theorems I, II, III imply that if a simple Lie group $\mathfrak{G}$ is contained in no one of the series A_n, B_n, C_n, D_n, then the diagram of the system $\Pi(\mathfrak{G})$ of its simple roots coincides with one of the diagrams (1) – (5) of Figure 5. (*The factor λ is determined uniquely by* 2 (3).) *Referring to the existence* of five distinct simple groups outside the series A_n, B_n, C_n, D_n, we state the following final theorem.

THEOREM IV. *All simple groups are represented by four infinite series A_n, B_n, C_n, D_n and five exceptional groups G_2, F_4, E_6, E_7, E_8. The systems of simple roots of the five exceptional groups are given, respectively, by the diagrams (1) – (5) of Figure 5.*

References

[1] B.L. van der Waerden. *Die Klassifikation der einfachen Lieschen Gruppen.* Math. Zeitschr.,**37** (1933), 446–462.

[2] H. Weyl. *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen.* Math. Zeitschr., **23** (1925), 271–309; **24** (1926), 328–395.

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