Thus let us define $\widetilde{y}_n$ by $y_n(x_1, \dots, x_n) = x_n + \widetilde{y}_n(x_1, \dots, x_n)$. Then we have

$$\exp(y_n(x_1, \dots, x_n)X_n) = \exp(\widetilde{y}_n(x_1, \dots, x_n)X_n)\exp(x_nX_n). \tag{1.132}$$

Since X_n is central, we have also

$$\exp(x_1X_1+\cdots+x_nX_n) = \exp(x_1X_1+\cdots+x_{n-1}X_{n-1})\exp(x_nX_n). \tag{1.133}$$

Substituting from (1.132) and (1.133) into (1.129), using (1.131), and canceling $\exp(x_nX_n)$ from both sides, we obtain

$$\begin{aligned}
&\exp(x_1X_1 + \cdots + x_{n-1}X_{n-1})\\
&= \exp((x_1 + \widetilde{y}_1)X_1)\exp((x_2 + \widetilde{y}_2(x_1))X_2)\\
&\quad \times \cdots \times \exp((x_{n-1} + \widetilde{y}_{n-1}(x_1, \dots, x_{n-2}))X_{n-1})\exp(\widetilde{y}_n(x_1, \dots, x_n)X_n).
\end{aligned}$$

The left side is independent of x_n, and hence so is the right side. Therefore $\widetilde{y}_n(x_1, \dots, x_n)$ is independent of x_n, and the proof of (1.130) for $i = n$ is complete.

Corollary 1.134. If N is a simply connected nilpotent analytic group, then any analytic subgroup of N is simply connected and closed.

PROOF. Let $\mathfrak{n}$ be the Lie algebra of N. Let M be an analytic subgroup of N, let $\mathfrak{m} \subseteq \mathfrak{n}$ be its Lie algebra, let $\widetilde{M}$ be the universal covering group of M, and let $\psi : \widetilde{M} \to M$ be the covering homomorphism. Assuming that M is not simply connected, let $\widetilde{m} \neq 1$ be in $\ker \psi$. Since exp is one-one onto for $\widetilde{M}$ by Theorem 1.127, we can find $X \in \mathfrak{m}$ with $\exp_{\widetilde{M}} X = \widetilde{m}$. Evidently $X \neq 0$. By (1.82) applied to ψ, $\exp_M X = 1$. By (1.82) applied to the inclusion of M into N, $\exp_N X = 1$. But this identity contradicts the assertion in Theorem 1.127 that exp is one-one for N. We conclude that M is simply connected. Since $\exp_M$ and $\exp_N$ are consistent, the image of $\mathfrak{m}$ under the diffeomorphism $\exp_N : \mathfrak{n} \to N$ is M, and hence M is closed.

17. Classical Semisimple Lie Groups

The classical semisimple Lie groups are specific closed linear groups that are connected and have semisimple Lie algebras listed in §8. Technically we have insisted that closed linear groups be closed subgroups of $GL(n, \mathbb{R})$

or $GL(n, \mathbb{C})$ for some n, but it will be convenient to allow closed subgroups of the group $GL(n, \mathbb{H})$ of nonsingular quaternion matrices as well.

The groups will be topologically closed because they are in each case the sets of common zeros of some polynomial functions in the entries. Most of the verification that the groups have particular linear Lie algebras as in §8 will be routine. It is necessary to make a separate calculation for the **special linear group**

$$SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) \mid \det g = 1\},$$

and this step was carried out in the Introduction; formula (0.10) and Proposition 0.11e allowed us to see that the linear Lie algebra of $SL(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{C})$.

In practice we use this result by combining it with a result about intersections: If G_1 and G_2 are closed linear groups with respective linear Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then the closed linear group $G_1 \cap G_2$ has linear Lie algebra $\mathfrak{g}_1 \cap \mathfrak{g}_2$. This fact follows immediately from the characterization in Proposition 0.14 of the linear Lie algebra as the set of all matrices X such that $\exp tX$ is in the corresponding group for all real t. Thus when "$\det g = 1$" appears as a defining condition for a closed linear group, the corresponding condition to impose for the linear Lie algebra is "$\operatorname{Tr} X = 0$."

The issue that tends to be more complicated is the connectedness of the given group. If we neglect to prove connectedness, we do not end up with the conclusion that the given group is semisimple, only that its identity component is semisimple.

To handle connectedness, we proceed in two steps, first establishing connectedness for certain compact examples and then proving in general that the number of components of the given group is the same as for a particular compact subgroup. We return to this matter at the end of this section.

We turn to a consideration of specific compact groups. Define

$$\begin{aligned} SO(n) &= \{g \in GL(n, \mathbb{R}) \mid g^*g = 1 \text{ and } \det g = 1\} \\ SU(n) &= \{g \in GL(n, \mathbb{C}) \mid g^*g = 1 \text{ and } \det g = 1\} \\ Sp(n) &= \{g \in GL(n, \mathbb{H}) \mid g^*g = 1\}. \end{aligned} \tag{1.135}$$

These are all closed linear groups, and they are compact by the Heine-Borel Theorem, their entries being bounded in absolute value by 1. The group $SO(n)$ is called the **rotation group**, and $SU(n)$ is called the **special**

unitary group. The group $Sp(n)$ is the **unitary group over the quaternions**. No determinant condition is imposed for $Sp(n)$. Artin [1957], pp. 151–158, gives an exposition of Dieudonné's notion of determinant for square matrices with entries from $\mathbb{H}$. The determinant takes real values ≥ 0, is multiplicative, is 1 on the identity matrix, and is 0 exactly for singular matrices. For the members of $Sp(n)$, the determinant is automatically 1.

Proposition 1.136. The groups $SO(n)$, $SU(n)$, and $Sp(n)$ are all connected for $n \geq 1$. The groups $SU(n)$ and $Sp(n)$ are all simply connected for $n \geq 1$, and the fundamental group of $SO(n)$ has order at most 2 for $n \geq 3$.

REMARK. Near the end of Chapter V, we shall see that the fundamental group of $SO(n)$ has order exactly 2 for $n \geq 3$.

PROOF. Consider $SO(n)$. For $n = 1$, this group is trivial and is therefore connected. For $n \geq 2$, $SO(n)$ acts transitively on the unit sphere in the space $\mathbb{R}^n$ of n-dimensional column vectors with entries from $\mathbb{R}$, and the isotropy subgroup at the n^{th} standard basis vector e_n is given in block form by

$$\begin{pmatrix} SO(n-1) & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the continuous map $g \mapsto ge_n$ of $SO(n)$ onto the unit sphere descends to a one-one continuous map of $SO(n)/SO(n-1)$ onto the unit sphere. Since $SO(n)/SO(n-1)$ is compact, this map is a homeomorphism. Consequently $SO(n)/SO(n-1)$ is connected. To complete the argument for connectivity of $SO(n)$, we induct on n, using the fact about topological groups that if H and G/H are connected, then G is connected.

For $SU(n)$, we argue similarly, replacing $\mathbb{R}$ by $\mathbb{C}$. The group $SU(1)$ is trivial and connected, and the action of $SU(n)$ on the unit sphere in $\mathbb{C}^n$ is transitive for $n \geq 2$. For $Sp(n)$, we argue with $\mathbb{H}$ in place of $\mathbb{R}$. The group $Sp(1)$ is the unit quaternions and is connected, and the action of $Sp(n)$ on the unit sphere in $\mathbb{H}^n$ is transitive for $n \geq 2$.

The assertions about fundamental groups follow from Corollary 1.98, the simple connectivity of $SU(1)$ and $Sp(1)$, and the fact that $SO(3)$ has fundamental group of order 2. This fact about $SO(3)$ follows from the simple connectivity of $SU(2)$ and the existence of a covering map $SU(2) \to SO(3)$. This covering map is the lift to analytic groups of the composition of the Lie algebra isomorphisms (1.4) and (1.3b).

It is clear from Proposition 1.136 and its remark that the linear Lie algebras of $SO(n)$ and $SU(n)$ are $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$, respectively. In the case of matrices with quaternion entries, we did not develop a theory of closed linear groups, but we can use the correspondence in §8 of n-by-n matrices over $\mathbb{H}$ with certain $2n$-by-$2n$ matrices over $\mathbb{C}$ to pass from $Sp(n)$ to complex matrices of size $2n$, then to the linear Lie algebra, and then back to $\mathfrak{sp}(n)$. In this sense the linear Lie algebra of $Sp(n)$ is $\mathfrak{sp}(n)$.

Taking into account the values of n in §8 for which these Lie algebras are semisimple, we conclude that $SO(n)$ is compact semisimple for $n \geq 3$, $SU(n)$ is compact semisimple for $n \geq 2$, and $Sp(n)$ is compact semisimple for $n \geq 1$.

Two families of related compact groups are

$$(1.137)\qquad \begin{aligned} O(n) &= \{g \in GL(n, \mathbb{R}) \mid g^* g = 1\} \\ U(n) &= \{g \in GL(n, \mathbb{C}) \mid g^* g = 1\}. \end{aligned}$$

These are the **orthogonal group** and the **unitary group**, respectively. The group $O(n)$ has two components; the Lie algebra is $\mathfrak{so}(n)$, and the identity component is $SO(n)$. The group $U(n)$ is connected by an argument like that in Proposition 1.136, and its Lie algebra is the reductive Lie algebra $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathbb{R}$.

Next we consider complex semisimple groups. According to §8, $\mathfrak{sl}(n, \mathbb{C})$ is semisimple for $n \geq 2$, $\mathfrak{so}(n, \mathbb{C})$ is semisimple for $n \geq 3$, and $\mathfrak{sp}(n, \mathbb{C})$ is semisimple for $n \geq 1$. Letting $J_{n,n}$ be as in §8, we define closed linear groups by

$$(1.138)\qquad \begin{aligned} SL(n, \mathbb{C}) &= \{g \in GL(n, \mathbb{C}) \mid \det g = 1\} \\ SO(n, \mathbb{C}) &= \{g \in SL(n, \mathbb{C}) \mid g^t g = 1\} \\ Sp(n, \mathbb{C}) &= \{g \in SL(2n, \mathbb{C}) \mid g^t J_{n,n} g = J_{n,n}\}. \end{aligned}$$

We readily check that their linear Lie algebras are $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, and $\mathfrak{sp}(n, \mathbb{C})$, respectively. Since $GL(n, \mathbb{C})$ is a complex Lie group and each of these Lie subalgebras of $\mathfrak{gl}(n, \mathbb{C})$ is closed under multiplication by i, Corollary 1.116 says that each of these closed linear groups G has the natural structure of a complex manifold in such a way that multiplication and inversion are holomorphic.

Proposition 1.139. Under the identification $M \mapsto Z(M)$ in (1.65),

$$Sp(n) \cong Sp(n, \mathbb{C}) \cap U(2n).$$

PROOF. From (1.65) we see that a $2n$-by-$2n$ complex matrix W is of the form $Z(M)$ if and only if

$$JW = \overline{W}J. \tag{1.140}$$

Let g be in $Sp(n)$. From $g^*g = 1$, we obtain $Z(g)^*Z(g) = 1$. Thus $Z(g)$ is in $U(2n)$. Also (1.140) gives $Z(g)^t JZ(g) = Z(g)^t\overline{Z(g)}J = (\overline{Z(g)^*Z(g)})J = J$, and hence $Z(g)$ is in $Sp(n,\mathbb{C})$.

Conversely suppose that W is in $Sp(n,\mathbb{C}) \cap U(2n)$. From $W^*W = 1$ and $W^tJW = J$, we obtain $J = W^t\bar{W}\bar{W}^{-1}JW = \overline{(W^*W)}\bar{W}^{-1}JW = \bar{W}^{-1}JW$ and therefore $\overline{W}J = JW$. By (1.140), $W = Z(g)$ for some quaternion matrix g. From $W^*W = 1$, we obtain $Z(g^*g) = Z(g)^*Z(g) = 1$ and $g^*g = 1$. Therefore g is in $Sp(n)$.

We postpone to the end of this section a proof that the groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, and $Sp(n,\mathbb{C})$ are connected for all n. We shall see that the proof of this connectivity reduces in the respective cases to the connectivity of $SU(n)$, $SO(n)$, and $Sp(n,\mathbb{C}) \cap U(2n)$, and this connectivity has been proved in Propositions 1.136 and 1.139. We conclude that $SL(n,\mathbb{C})$ is semisimple for $n \geq 2$, $SO(n,\mathbb{C})$ is semisimple for $n \geq 3$, and $Sp(n,\mathbb{C})$ is semisimple for $n \geq 1$.

The groups $SO(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$ have interpretations in terms of bilinear forms. The group $SO(n,\mathbb{C})$ is the subgroup of matrices in $SL(n,\mathbb{C})$ preserving the symmetric bilinear form on $\mathbb{C}^n \times \mathbb{C}^n$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1y_1 + \cdots + x_ny_n,$$

while the group $Sp(n,\mathbb{C})$ is the subgroup of matrices in $SL(2n,\mathbb{C})$ preserving the alternating bilinear form on $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix} \right\rangle = x_1y_{n+1} + \cdots + x_ny_{2n} - x_{n+1}y_1 - \cdots - x_{2n}y_n.$$

Finally we consider noncompact noncomplex semisimple groups. With notation $I_{m,n}$ and $J_{n,n}$ as in §8, the definitions are

$$\begin{aligned}
SL(n,\mathbb{R}) &= \{g \in GL(n,\mathbb{R}) \mid \det g = 1\}\\
SL(n,\mathbb{H}) &= \{g \in GL(n,\mathbb{H}) \mid \det g = 1\}\\
SO(m,n) &= \{g \in SL(m+n,\mathbb{R}) \mid g^* I_{m,n} g = I_{m,n}\}\\
SU(m,n) &= \{g \in SL(m+n,\mathbb{C}) \mid g^* I_{m,n} g = I_{m,n}\}\\
Sp(m,n) &= \{g \in GL(m+n,\mathbb{H}) \mid g^* I_{m,n} g = I_{m,n}\}\\
Sp(n,\mathbb{R}) &= \{g \in SL(2n,\mathbb{R}) \mid g^t J_{n,n} g = J_{n,n}\}\\
SO^*(2n) &= \{g \in SU(n,n) \mid g^t I_{n,n} J_{n,n} g = I_{n,n} J_{n,n}\}.
\end{aligned} \tag{1.141}$$

Some remarks are in order about particular groups in this list. For $SL(n,\mathbb{H})$ and $Sp(m,n)$, the prescription at the end of §8 allows us to replace the realizations in terms of quaternion matrices by realizations in terms of complex matrices of twice the size. The realization of $SL(n,\mathbb{H})$ with complex matrices avoids the notion of determinant of a quaternion matrix that was mentioned before the statement of Proposition 1.136; the isomorphic group of complex matrices is

$$SU^*(2n) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SL(2n,\mathbb{C}) \right\}.$$

The groups $SO(m,n)$, $SU(m,n)$, and $Sp(m,n)$ are isometry groups of Hermitian forms. In more detail the group

$$O(m,n) = \{g \in GL(m+n,\mathbb{R}) \mid g^* I_{m,n} g = I_{m,n}\}$$

is the group of real matrices of size $m+n$ preserving the symmetric bilinear form on $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1 y_1 + \cdots + x_m y_m - x_{m+1} y_{m+1} - \cdots - x_{m+n} y_{m+n},$$

and $SO(m,n)$ is the subgroup of members of $O(m,n)$ of determinant 1. The group

$$U(m,n) = \{g \in GL(m+n,\mathbb{C}) \mid g^* I_{m,n} g = I_{m,n}\}$$

is the group of complex matrices of size $m+n$ preserving the Hermitian form on $\mathbb{C}^{m+n} \times \mathbb{C}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1 \overline{y_1} + \cdots + x_m \overline{y_m} - x_{m+1} \overline{y_{m+1}} - \cdots - x_{m+n} \overline{y_{m+n}},$$

and $SU(m,n)$ is the subgroup of members of $U(m,n)$ of determinant 1. The group $Sp(m,n)$ is the group of quaternion matrices of size $m+n$ preserving the Hermitian form on $\mathbb{H}^{m+n}\times\mathbb{H}^{m+n}$ given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix} \right\rangle = x_1\overline{y_1}+\cdots+x_m\overline{y_m}-x_{m+1}\overline{y_{m+1}}-\cdots-x_{m+n}\overline{y_{m+n}},$$

with no condition needed on the determinant.

The linear Lie algebras of the closed linear groups in (1.141) are given in a table in Example 3 of §8, and the table in §8 tells which values of m and n lead to semisimple Lie algebras. It will be a consequence of results below that all the closed linear groups in (1.141) are topologically connected except for $SO(m,n)$. In the case of $SO(m,n)$, one often works with the identity component $SO(m,n)_0$ in order to have access to the full set of results about semisimple groups in later chapters.

Let us now address the subject of connectedness in detail. We shall work with a closed linear group of complex matrices that is closed under adjoint and is defined by polynomial equations. We begin with a lemma.

Lemma 1.142. Let $P:\mathbb{R}^n\to\mathbb{R}$ be a polynomial, and suppose $(a_1,\dots,a_n)$ has the property that $P(e^{ka_1},\dots,e^{ka_n})=0$ for all integers $k\geq 0$. Then $P(e^{ta_1},\dots,e^{ta_n})=0$ for all real t.

PROOF. A monomial $cx_1^{l_1}\cdots x_n^{l_n}$, when evaluated at $(e^{ta_1},\dots,e^{ta_n})$, becomes $ce^{t\sum a_il_i}$. Collecting terms with like exponentials, we may assume that we have an expression $\sum_{j=1}^N c_je^{tb_j}$ that vanishes whenever t is an integer ≥ 0. We may further assume that all c_j are nonzero and that $b_1<b_2<\cdots<b_N$. We argue by contradiction and suppose $N>0$. Multiplying by e^{-tb_N} and changing notation, we may assume that $b_N=0$. We pass to the limit in the expression $\sum_{j=1}^N c_je^{tb_j}$ as t tends to $+\infty$ through integer values, and we find that $c_N=0$, contradiction.

Proposition 1.143. Let $G\subseteq GL(n,\mathbb{C})$ be a closed linear group that is the common zero locus of some set of real-valued polynomials in the real and imaginary parts of the matrix entries, and let $\mathfrak{g}$ be its linear Lie algebra. Suppose that G is closed under adjoints. Let K be the group $G\cap U(n)$, and let $\mathfrak{p}$ be the subspace of Hermitian matrices in $\mathfrak{g}$. Then the map $K\times\mathfrak{p}\to G$ given by $(k,X)\mapsto ke^X$ is a homeomorphism onto.

PROOF. For $GL(n, \mathbb{C})$, the map

$$U(n) \times \{\text{Hermitian matrices}\} \to GL(n, \mathbb{C})$$

given by $(k, X) \mapsto ke^X$ is known to be a homeomorphism; see Chevalley [1946], pp. 14–15. The inverse map is the **polar decomposition** of $GL(n, \mathbb{C})$.

Let g be in G, and let $g = ke^X$ be the polar decomposition of g within $GL(n, \mathbb{C})$. To prove the proposition, we have only to show that k is in G and that X is in the linear Lie algebra $\mathfrak{g}$ of G.

Taking adjoints, we have $g^* = e^X k^{-1}$ and therefore $g^* g = e^{2X}$. Since G is closed under adjoints, e^{2X} is in G. By assumption, G is the zero locus of some set of real-valued polynomials in the real and imaginary parts of the matrix entries. Let us conjugate matters so that e^{2X} is diagonal, say $2X = \operatorname{diag}(a_1, \ldots, a_n)$ with each a_j real. Since e^{2X} and its integral powers are in G, the transformed polynomials vanish at

$$(e^{2X})^k = \operatorname{diag}(e^{ka_1}, \ldots, e^{ka_n})$$

for every integer k. By Lemma 1.142 the transformed polynomials vanish at $\operatorname{diag}(e^{ta_1}, \ldots, e^{ta_n})$ for all real t. Therefore e^{tX} is in G for all real t. It follows from the definition of $\mathfrak{g}$ that X is in $\mathfrak{g}$. Since e^X and g are then in G, k is in G. This completes the proof.

Proposition 1.143 says that G is connected if and only if K is connected. To decide which of the groups in (1.138) and (1.141) are connected, we therefore compute K for each group. In the case of the groups of quaternion matrices, we compute K by converting to complex matrices, intersecting with the unitary group, and transforming back to quaternion matrices. The results are in (1.144). In the K column of (1.144), the notation $S(\cdot)$ means

(1.144)

G	K up to isomorphism
$SL(n, \mathbb{C})$	$SU(n)$
$SO(n, \mathbb{C})$	$SO(n)$
$Sp(n, \mathbb{C})$	$Sp(n)$ or $Sp(n, \mathbb{C}) \cap U(2n)$
$SL(n, \mathbb{R})$	$SO(n)$
$SL(n, \mathbb{H})$	$Sp(n)$
$SO(m, n)$	$S(O(m) \times O(n))$
$SU(m, n)$	$S(U(m) \times U(n))$
$Sp(m, n)$	$Sp(m) \times Sp(n)$
$Sp(n, \mathbb{R})$	$U(n)$
$SO^*(2n)$	$U(n)$

the determinant-one subgroup of $(\cdot)$. By Propositions 1.136 and 1.139 and the connectedness of $U(n)$, we see that all the groups in the K column are connected except for $S(O(m) \times O(n))$. Using Proposition 1.143, we arrive at the following conclusion.

Proposition 1.145. All the classical groups $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SL(n,\mathbb{H})$, $SU(m,n)$, $Sp(m,n)$, $Sp(n,\mathbb{R})$, and $SO^*(2n)$ are connected. The group $SO(m,n)$ has two components if $m > 0$ and $n > 0$.

18. Problems

1. Verify that Example 12a in §1 is nilpotent and that Example 12b is split solvable.

2. For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ any nonsingular matrix over $\mathbb{k}$, let $\mathfrak{g}_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}$ be the 3-dimensional algebra over $\mathbb{k}$ with basis X, Y, Z satisfying

$$\begin{aligned} [X,Y] &= 0 \\ [X,Z] &= \alpha X + \beta Y \\ [Y,Z] &= \gamma X + \delta Y. \end{aligned}$$

   (a) Show that $\mathfrak{g}_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}$ is a *Lie* algebra by showing that $X \leftrightarrow \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$, $Y \leftrightarrow \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$, $Z \leftrightarrow -\begin{pmatrix} \alpha&\gamma&0 \\ \beta&\delta&0 \\ 0&0&0 \end{pmatrix}$ gives an isomorphism with a Lie algebra of matrices.

   (b) Show that $\mathfrak{g}_{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)}$ is solvable but not nilpotent.

   (c) Let $\mathbb{k} = \mathbb{R}$. Take $\delta = 1$ and $\beta = \gamma = 0$. Show that the various Lie algebras $\mathfrak{g}_{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)}$ for $\alpha > 1$ are mutually nonisomorphic. (Therefore for $k = \mathbb{R}$ that there are uncountably many nonisomorphic solvable real Lie algebras of dimension 3.)

3. Let

$$\mathfrak{s}(n,\mathbb{k}) = \left\{ X \in \mathfrak{gl}(n,\mathbb{k}) \;\middle|\; X = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\}.$$