

1. Classical Groups I

The elements of the groups defined in this chapter are matrices with entries in one of the three fields:

\mathbf{R} the field of *real numbers*,
 \mathbf{C} the field of *complex numbers*,
 \mathbf{H} the field of *quaternions*.

Note that \mathbf{H} , the field of quaternions (or hamiltonians), is *not* commutative. The quaternions will be examined in great detail, along with the *octonions* \mathbf{O} (or *Cayley numbers*), in Chapter 6 on normed algebras. For the purposes of this chapter, a rudimentary knowledge of \mathbf{H} is all that is presupposed. Consult Problem 6 for the multiplication rules for quaternions.

Let $M_n(\mathbf{R})$, $M_n(\mathbf{C})$, and $M_n(\mathbf{H})$ denote the algebras of $n \times n$ matrices with entries in \mathbf{R} , \mathbf{C} , and \mathbf{H} respectively. Represent elements x of \mathbf{R}^n , \mathbf{C}^n , and \mathbf{H}^n as column n -tuples. Then each matrix A determines a linear transformation or endomorphism $x \mapsto Ax$ by letting the matrix A act on the left of the column vector x , at least in the real and complex case. Special consideration is necessary for the quaternionic case since \mathbf{H} is not commutative. In order for the map $A : \mathbf{H}^n \rightarrow \mathbf{H}^n$ (defined by A acting on x on left) to be \mathbf{H} -linear, we are forced to let the scalars \mathbf{H} act on the \mathbf{H} -vector space \mathbf{H}^n on the right!

Although it will be convenient to consider both *right \mathbf{H} -vector spaces* (where the scalars \mathbf{H} act on the right of the vectors) and *left \mathbf{H} -vector spaces* (where the scalars \mathbf{H} act on the left of the vectors), the space \mathbf{H}^n of

column n -tuples will always be considered as a right \mathbf{H} -vector space. Then we have

$$(1.1) \quad M_n(\mathbf{R}) \cong \text{End}_{\mathbf{R}}(\mathbf{R}^n),$$

$$(1.2) \quad M_n(\mathbf{C}) \cong \text{End}_{\mathbf{C}}(\mathbf{C}^n),$$

$$(1.3) \quad M_n(\mathbf{H}) \cong \text{End}_{\mathbf{H}}(\mathbf{H}^n).$$

Here $\text{End}_F(V)$ denotes the F -linear maps from a vector space V , with scalar field F , into itself. If V is a real vector space, then $\text{End}_{\mathbf{R}}V$ is naturally a real algebra (associative and with unit). If V is a complex vector space, then $\text{End}_{\mathbf{C}}V$ is naturally a complex algebra (associative and with unit) but may also be considered as a real algebra, which is convenient for some immediate purposes. Finally, if V is a right quaternionic vector space, then $\text{End}_{\mathbf{H}}(V)$ is naturally a *real* algebra—in fact, a real subalgebra of the algebra $\text{End}_{\mathbf{R}}(V)$. There is no canonical way to make $\text{End}_{\mathbf{H}}(V)$ into even a quaternionic vector space (right or left), much less a “quaternionic” algebra (see Problem 7).

THE GENERAL LINEAR GROUPS

The group of units, or invertible elements, in the matrix algebra $M_n(F)$ is called the F -general linear group for $F \equiv \mathbf{R}, \mathbf{C}$, or \mathbf{H} and is denoted by $\text{GL}(n, \mathbf{R})$, $\text{GL}(n, \mathbf{C})$, or $\text{GL}(n, \mathbf{H})$, respectively. If the group of units, or invertible elements, in $\text{End}_F(V)$ is denoted by $\text{GL}_F(V)$, then

$$(1.1') \quad \text{GL}(n, \mathbf{R}) \cong \text{GL}_{\mathbf{R}}(\mathbf{R}^n),$$

$$(1.2') \quad \text{GL}(n, \mathbf{C}) \cong \text{GL}_{\mathbf{C}}(\mathbf{R}^n),$$

$$(1.3') \quad \text{GL}(n, \mathbf{H}) \cong \text{GL}_{\mathbf{H}}(\mathbf{H}^n).$$

In the quaternion case, there is another important group, larger than the \mathbf{H} -general linear group $\text{GL}(n, \mathbf{H})$, which we will call the *enhanced \mathbf{H} -general linear group*. First note that the \mathbf{H} -general linear group $\text{GL}(n, \mathbf{H})$ (which acts on the left) consists entirely of \mathbf{H} -linear maps. However, right multiplication by a scalar $\lambda \in \mathbf{H}$, denoted R_λ , is not necessarily \mathbf{H} -linear. In fact, R_λ is \mathbf{H} -linear if and only if λ commutes with all scalars $\mu \in \mathbf{H}$

because $R_\lambda(x\mu) = x\mu\lambda$, while $R_\lambda(x)\mu = x\lambda\mu$. The reader should confirm that λ commutes with all $\mu \in \mathbf{H}$ if and only if $\lambda \in \mathbf{R} \subset \mathbf{H}$. Thus, R_λ is \mathbf{H} -linear if and only if $\lambda \in \mathbf{R} \subset \mathbf{H}$. Let \mathbf{H}^* denote the group of right multiplications by nonzero scalars. Then \mathbf{H}^* is not a subgroup of $\mathrm{GL}(n, \mathbf{H})$. However, both are contained in the algebra $\mathrm{End}_{\mathbf{R}}(\mathbf{H}^n)$ of \mathbf{R} -linear maps. As noted above, the intersection $\mathrm{GL}(n, \mathbf{H}) \cap \mathbf{H}^*$ equals \mathbf{R}^* the group of real nonzero multiples of the identity.

The *enhanced \mathbf{H} -general linear group*, denoted $\mathrm{GL}(n, \mathbf{H}) \cdot \mathbf{H}^*$, is defined to be the image of $\mathrm{GL}(n, \mathbf{H}) \times \mathbf{H}^*$ in $\mathrm{End}_{\mathbf{R}}(\mathbf{H}^n)$ via the map sending the pair (A, λ) to $L_A \cdot R_\lambda$, where \cdot denotes multiplication in the algebra $\mathrm{End}_{\mathbf{R}}(\mathbf{H}^n)$, i.e., composition. Thus, the following sequence of groups is exact:

$$(1.4) \quad 1 \rightarrow \mathbf{R}^* \rightarrow \mathrm{GL}(n, \mathbf{H}) \times \mathbf{H}^* \rightarrow \mathrm{GL}(n, \mathbf{H}) \cdot \mathbf{H}^* \rightarrow 1$$

with $\mathrm{GL}(n, \mathbf{H}) \cdot \mathbf{H}^* \subset \mathrm{End}_{\mathbf{R}}(\mathbf{H}^n)$. Note that the larger group $\mathrm{GL}(n, \mathbf{H}) \cdot \mathbf{H}^*$, as well as the smaller group $\mathrm{GL}(n, \mathbf{H})$, maps quaternion lines to quaternion lines.

Given $A \in M_n(\mathbf{R})$, the real determinant of A will be denoted $\det_{\mathbf{R}} A$. Similarly, $\det_{\mathbf{C}} A$ denotes the complex determinant of $A \in M_n(\mathbf{C})$. The lack of commutativity for \mathbf{H} eliminates the possibility of any useful notion of “quaternionic determinant.” Of course,

$$(1.5) \quad \begin{aligned} &\mathrm{GL}(n, \mathbf{R}) = \{A \in M_n(\mathbf{R}) : \det_{\mathbf{R}} A \neq 0\}, \\ &\text{and} \\ &\mathrm{GL}(n, \mathbf{C}) = \{A \in M_n(\mathbf{C}) : \det_{\mathbf{C}} A \neq 0\}. \end{aligned}$$

The group

$$(1.6) \quad \mathrm{GL}^+(n, \mathbf{R}) = \{A \in M_n(\mathbf{R}) : \det_{\mathbf{R}} A > 0\}$$

is called the *orientation-preserving general linear group*.

In both the real and the complex case, we have a *special linear group*, defined by

$$(1.7) \quad \mathrm{SL}(n, \mathbf{R}) \equiv \{A \in M_n(\mathbf{R}) : \det_{\mathbf{R}} A = 1\},$$

$$(1.8) \quad \mathrm{SL}(n, \mathbf{C}) \equiv \{A \in M_n(\mathbf{C}) : \det_{\mathbf{C}} A = 1\}.$$

Since there is no quaternion determinant, if we proceed in exact analogy with the real or the complex case, the special quaternion linear group

does not exist. However, it is useful to retain the notation $SL(n, \mathbf{H})$ by employing the real determinant. Let

$$(1.9) \quad SL(n, \mathbf{H}) \equiv \{A \in GL(n, \mathbf{H}) : \det_{\mathbf{R}} A = 1\}$$

denote the *special quaternion linear group*.

GROUPS DEFINED BY BILINEAR FORMS

Some very interesting groups are best defined as subgroups of the groups defined above that fix a certain bilinear form.

\mathbf{R} -symmetric

The *orthogonal group* $O(p, q)$ with signature p, q is defined to be the subgroup of $GL(n, \mathbf{R})$ ($n \equiv p + q$) that fixes the *standard \mathbf{R} -symmetric form*

$$(1.10) \quad \varepsilon(x, y) \equiv x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_n y_n.$$

That is,

$$O(p, q) \equiv \{A \in GL(n, \mathbf{R}) : \varepsilon(Ax, Ay) = \varepsilon(x, y) \text{ for all } x, y \in \mathbf{R}^n\}.$$

\mathbf{R} -skew (or symplectic)

The *real symplectic group* $Sp(n, \mathbf{R})$ is defined to be the subgroup of $GL(2n, \mathbf{R})$ that fixes the *standard \mathbf{R} -symplectic (or \mathbf{R} -skew) form*

$$(1.11) \quad \varepsilon \equiv dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n},$$

or

$$(1.11') \quad \varepsilon(x, y) \equiv x_1 y_2 - x_2 y_1 + \cdots + x_{2n-1} y_{2n} - x_{2n} y_{2n-1}.$$

That is,

$$Sp(n, \mathbf{R}) \equiv \{A \in GL(2n, \mathbf{R}) : \varepsilon(Ax, Ay) = \varepsilon(x, y) \text{ for all } x, y \in \mathbf{R}^{2n}\}.$$

C-symmetric

The *complex orthogonal group* $O(n, \mathbb{C})$ is defined to be the subgroup of $GL(n, \mathbb{C})$ that fixes the *standard C-symmetric form*

$$(1.12) \quad \varepsilon(z, w) \equiv z_1 w_1 + \cdots + z_n w_n.$$

C-skew (or symplectic)

The *complex symplectic group* $Sp(n, \mathbb{C})$ is defined to be the subgroup of $GL(2n, \mathbb{C})$ that fixes the *standard C-symplectic (or C-skew) form*

$$(1.13) \quad \varepsilon \equiv dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n},$$

or

$$(1.13') \quad \varepsilon(z, w) \equiv z_1 w_2 - z_2 w_1 + \cdots + z_{2n-1} w_{2n} - z_{2n} w_{2n-1}.$$

C-hermitian (symmetric)

The *complex unitary group* $U(p, q)$ with *signature* p, q is defined to be the subgroup of $GL(n, \mathbb{C})$ ($n \equiv p + q$) that fixes the *standard C-hermitian symmetric form*

$$(1.14) \quad \varepsilon(z, w) \equiv z_1 \bar{w}_1 + \cdots + z_p \bar{w}_p - z_{p+1} \bar{w}_{p+1} - \cdots - z_n \bar{w}_n.$$

Remark 1.15. $i\varepsilon(z, w)$ is called the *standard C-hermitian skew form*. Note that the group that fixes $i\varepsilon$ is just the same group $U(p, q)$ that fixes ε . This contrasts sharply with the quaternion case.

H-hermitian symmetric

The *hyper-unitary group* $HU(p, q)$ with *signature* p, q is defined to be the subgroup of $GL(n, \mathbb{H})$ ($n \equiv p + q$) that fixes the *standard H-hermitian symmetric form*

$$(1.16) \quad \varepsilon(x, y) \equiv \bar{x}_1 y_1 + \cdots + \bar{x}_p y_p - \bar{x}_{p+1} y_{p+1} - \cdots - \bar{x}_n y_n.$$

Note: $\varepsilon(x, y)$ is *H-hermitian*. This means that ε is additive in both variables x and y , and $\varepsilon(x\lambda, y) = \bar{\lambda}\varepsilon(x, y)$, $\varepsilon(x, y\lambda) = \varepsilon(x, y)\lambda$ for all scalars $\lambda \in \mathbb{H}$. Also note that xy is not *H-linear* in x . In fact, there is no standard *H-symmetric* or *H-skew* form (see Problem 8).

Remark. The group $\text{HU}(p, q)$ is usually denoted “ $\text{Sp}(p, q)$ ” and called the “symplectic group.”

H-hermitian skew

The *skew H-unitary group* $\text{SK}(n, \mathbf{H})$, or $\text{SK}(n)$, is defined to be the subgroup of $\text{GL}(n, \mathbf{H})$ that fixes the *standard H-hermitian skew form*

$$(1.17) \quad \varepsilon(x, y) \equiv \bar{x}_1 i y_1 + \cdots + \bar{x}_n i y_n.$$

Remark. This i is the quaternion i (see Problem 6). In Chapter 2, we shall see that if the i occurring in (1.17) is replaced by any unit imaginary quaternion $u \in S^2 \subset \text{Im } \mathbf{H}$, then the new form ε' differs from the old form ε by a coordinate change, i.e., an element of $\text{GL}(n, \mathbf{H})$.

Table 1.18. The groups defined by bilinear forms

	symmetric ϵ	skew ϵ	hermitian symmetric ϵ	hermitian skew ϵ
R	$O(p, q)$	$\text{Sp}(n, \mathbf{R})$		
C	$O(n, \mathbf{C})$	$\text{Sp}(n, \mathbf{C})$	$U(p, q)$	$U(p, q)$
H			$\text{HU}(p, q)$	$\text{SK}(n, \mathbf{H})$

OTHER MISCELLANEOUS GROUPS

The subgroups defined by requiring either $\det_{\mathbf{R}}$ or $\det_{\mathbf{C}}$ to be equal to one can also be defined by requiring that an n -form be fixed. The skew n -form

$$(1.19) \quad dx \equiv dx_1 \wedge \cdots \wedge dx_n$$

on \mathbf{R}^n is called the *standard volume form on \mathbf{R}^n* , while the skew n -form

$$(1.19') \quad dz \equiv dz_1 \wedge \cdots \wedge dz_n$$

on \mathbf{C}^n is called the *standard complex volume form on \mathbf{C}^n* . The volume form transforms, under a coordinate change, by multiplication by the determinant:

$$A^* dx = (\det_{\mathbf{R}} A) dx \quad \text{for all } A \in \text{End}_{\mathbf{R}} V$$

and

$$B^*dz = (\det_{\mathbf{C}} B) dz \quad \text{for all } B \in \text{End}_{\mathbf{C}} V.$$

Here A^* denotes the *dual* (or *pull back*) map associated with A , which is defined by $(A^*\alpha)(u) = \alpha(Au)$ if α is a form of degree one and by $A^*(\alpha^1 \wedge \cdots \wedge \alpha^k) = A^*\alpha^1 \wedge \cdots \wedge A^*\alpha^k$ if $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ is the simple product of degree one forms. This provides the most elegant definition of the determinant. Frequently, this is also the most useful. For example, see Problem 4. This definition gives

$$(1.20) \quad \text{SL}(n, \mathbf{R}) = \{A \in \text{GL}(n, \mathbf{R}) : A^*dx = dx\},$$

$$(1.20') \quad \text{SL}(n, \mathbf{C}) = \{A \in \text{GL}(n, \mathbf{C}) : A^*dz = dz\}.$$

The *special orthogonal group with signature p, q* is defined by

$$(1.21) \quad \text{SO}(p, q) \equiv \{A \in O(p, q) : \det_{\mathbf{R}} A = 1\}.$$

The *special complex orthogonal group* is defined by

$$(1.22) \quad \text{SO}(n, \mathbf{C}) \equiv \{A \in O(n, \mathbf{C}) : \det_{\mathbf{C}} A = 1\}.$$

The *special unitary group* is defined by

$$(1.23) \quad \text{SU}(p, q) \equiv \{A \in U(p, q) : \det_{\mathbf{C}} A = 1\}.$$

The various other possibilities do not lead to new groups. This is a consequence of the facts presented below—see (1.24), (1.25), (1.26), Lemma 1.28, (1.29), and (1.30).

Consult Problem 5 for proofs of the following:

$$(1.24) \quad \text{if } A \in \text{Sp}(n, \mathbf{R}), \text{ then } \det_{\mathbf{R}} A = 1;$$

and

$$(1.25) \quad \text{if } A \in \text{Sp}(n, \mathbf{C}), \text{ then } \det_{\mathbf{C}} A = 1.$$

Forgetting the complex structure on \mathbf{C}^n , the complex vector space \mathbf{C}^n becomes a real vector space of dimension $2n$. This embeds the algebra $\text{End}_{\mathbf{C}}(\mathbf{C}^n)$ of complex linear maps into the algebra $\text{End}_{\mathbf{R}}(\mathbf{C}^n)$ of all real linear maps. Thus, for $a \in M_n(\mathbf{C})$, the real determinant $\det_{\mathbf{R}} A$ has meaning as well as $\det_{\mathbf{C}} A$. See Problem 4 for a proof of the result:

$$(1.26) \quad \text{if } A \in M_n(\mathbf{C}), \text{ then } \det_{\mathbf{R}} A = |\det_{\mathbf{C}} A|^2.$$

The quaternion vector space \mathbf{H}^n can be considered as a complex vector space in a variety of natural ways (more precisely, a 2-sphere S^2 of natural ways). Let $\text{Im } \mathbf{H}$ denote the real hyperplane in \mathbf{H} with normal $1 \in \mathbf{H}$. Let S^2 denote the unit sphere in $\text{Im } \mathbf{H}$. Then, for each $u \in S^2$, $u^2 = -u\bar{u} = -|u|^2 = -1$. Therefore, right multiplication by u , defined by

$$R_u x \equiv xu \quad \text{for all } x \in \mathbf{H}^n,$$

is a *complex structure* on \mathbf{H}^n ; that is, $R_u^2 = -1$. This property enables one to define a complex scalar multiplication on \mathbf{H}^n by $(a + bi)x \equiv (a + bR_u)(x)$ for all $a, b \in \mathbb{R}$ and all $x \in \mathbf{H}^n$, where $i^2 = -1$. Note that $\text{End}_{\mathbf{H}}(\mathbf{H}^n) \subset \text{End}_{\mathbb{C}}(\mathbf{H}^n)$ for each of the complex structures R_u on \mathbf{H}^n , where $u \in S^2 \subset \text{Im } \mathbf{H}$. Choosing a complex basis for \mathbf{H}^n provides a complex linear isomorphism $\mathbf{H}^n \cong \mathbb{C}^{2n}$. Sometimes it is convenient to select this complex basis as follows. Let $\mathbf{C}(u)$ denote the complex line containing 1 in each of the axis subspaces $\mathbf{H} \subset \mathbf{H}^n$. Thus, $\mathbf{C}(u)$ is the real span of 1 and u . Let $\mathbf{C}(u)^\perp$ denote the complex line orthogonal to $\mathbf{C}(u)$ in $\mathbf{H} \subset \mathbf{H}^n$. Then

$$(1.27) \quad \mathbf{H}^n \cong [\mathbf{C}(u) \oplus \mathbf{C}(u)^\perp]^n \cong \mathbb{C}^{2n}.$$

Assume the complex structure on \mathbf{H}^n has been fixed, say R_i , then as noted above $\text{End}_{\mathbf{H}}(\mathbf{H}^n) \subset \text{End}_{\mathbb{C}}(\mathbb{C}^{2n})$. Moreover, given $A \in \text{End}_{\mathbb{C}}(\mathbb{C}^{2n})$, one can show that

$$A \in \text{End}_{\mathbf{H}}(\mathbf{H}^n) \text{ if and only if } AR_j = R_j A.$$

This is a useful characterization of the subspace $\text{End}_{\mathbf{H}}(\mathbf{H}^n)$ of $\text{End}_{\mathbb{C}}(\mathbb{C}^{2n})$.

Lemma 1.28. *For each complex structure R_u on \mathbf{H}^n (determined by right multiplication by a unit imaginary quaternion $u \in S^2 \subset \text{Im } \mathbf{H}$) and for each $A \in M_n(\mathbf{H})$ the complex determinant $\det_{\mathbb{C}} A$ is the positive square root of $\det_{\mathbb{R}} A$, independent of the complex structure R_u .*

Proof: First, we show that the complex determinant of $A \in M_n(\mathbf{H})$ is real for all $A \in M_n(\mathbf{H})$. We will give the proof for the particular complex structure R_i . Consider the case $n = 1$. Let $e_0 \equiv 1, e_1 \equiv i, e_2 \equiv j$, and $e_3 \equiv k$ denote the standard real basis for the quaternions \mathbf{H} . Let $\omega^0, \omega^1, \omega^2, \omega^3$ denote the standard dual basis. Then

$$dz^1 = \omega^0 + i\omega^1, \quad dz^2 = \omega^2 - i\omega^3$$

is a basis for the complex forms of type 1,0 on $\mathbf{H} \cong \mathbb{C}^2$ (with complex structure R_i). Note that $R_j^*(dz^1) = -d\bar{z}^2$ and $R_j^*(dz^2) = d\bar{z}^1$, so that $R_j(dz^1 \wedge dz^2) = d\bar{z}^1 \wedge d\bar{z}^2$.

Now

$$R_j^* A^* (dz^1 \wedge dz^2) = R_j^* (\det_{\mathbf{C}} A \, dz^1 \wedge dz^2) = \det_{\mathbf{C}} A \, d\bar{z}^1 \wedge d\bar{z}^2,$$

while

$$A^* R_j^* (dz^1 \wedge dz^2) = A^* (d\bar{z}^1 \wedge d\bar{z}^2) = \overline{\det_{\mathbf{C}} A} \, d\bar{z}^1 \wedge d\bar{z}^2.$$

Therefore, $\det_{\mathbf{C}} A \in \mathbf{R}$ is real since $AR_j = R_j A$. The proof, for $n > 1$, that $\det_{\mathbf{C}} A \in \mathbf{R}$ for all $A \in M_n(\mathbf{H})$ is similar and omitted. Because of (1.26), it remains to show that

$$\det_{\mathbf{C}} A > 0 \quad \text{if } A \in \text{GL}(n, \mathbf{H}).$$

Since $\det_{\mathbf{C}} I = 1$ and $\text{GL}(n, \mathbf{H})$ is connected (Problem 3), the set $\{\det_{\mathbf{C}} A : A \in \text{GL}(n, \mathbf{H})\}$ is a connected subset of $\mathbf{R} - \{0\}$ containing 1, and hence it is contained in \mathbf{R}^+ . ■

For elements of the subgroups $\text{HU}(p, q)$ and $\text{SK}(n, \mathbf{H})$ of $\text{GL}(n, \mathbf{H})$, the real determinant is already equal to one (and hence by Lemma 1.28 all the various complex determinants are also equal to one). That is,

$$(1.29) \quad \det_{\mathbf{R}} A = 1 \text{ if } A \in \text{HU}(p, q);$$

$$(1.30) \quad \det_{\mathbf{R}} A = 1 \text{ if } A \in \text{SK}(n, \mathbf{H}).$$

Both of these facts follow from (1.24), since both $\text{HU}(p, q)$ and $\text{SK}(n, \mathbf{H})$ are contained in $\text{Sp}(2n, \mathbf{R})$ for a suitable choice of coordinates. For example, if A fixes the ε defined by (1.16), i.e., $A \in \text{HU}(p, q)$, then A fixes the real valued skew form $\text{Re } i\varepsilon(x, y)$, which under a coordinate change is the symplectic form given by (1.11'). The details are provided in the next chapter—see Lemma 2.80 and Equation (2.91).

In the quaternion case, there is always the option of enlarging the group by utilizing right scalar multiplications. Recall (1.4) how the group $\text{GL}(n, \mathbf{H}) \cdot \mathbf{H}^*$ is an enhancement of the quaternionic general linear group $\text{GL}(n, \mathbf{H})$. For another example, consider the *enhanced hyper-unitary group* (perhaps a better name is the *quaternionic unitary group*). This group is denoted $\text{HU}(p, q) \cdot \text{HU}(1)$ and defined to be the subgroup of $\text{End}_{\mathbf{R}}(\mathbf{H}^n)$ generated by letting $\text{HU}(p, q)$ act on \mathbf{H}^n on the left and the unit scalars $\text{HU}(1) \cong S^3$ act on \mathbf{H}^n on the right. Since

$$(1.31) \quad 1 \longrightarrow \mathbf{Z}_2 \longrightarrow \text{HU}(p, q) \times \text{HU}(1) \xrightarrow{x} \text{End}_{\mathbf{R}}(\mathbf{H}^n)$$

is exact, where $\mathbf{Z}_2 = \{1, -1\}$ and where $\mathrm{HU}(1) \cong \{R_y : y \in S^3 \subset \mathbf{H}\}$, it follows that

$$(1.32) \quad \mathrm{HU}(p, q) \cdot \mathrm{HU}(1) \cong (\mathrm{HU}(p, q) \times \mathrm{HU}(1)) / \mathbf{Z}_2.$$

See Problem 3.15 for more information about the quaternionic unitary group $\mathrm{HU}(p, q) \cdot \mathrm{HU}(1)$. For example, this group fixes a 4-form $\Phi \in \Lambda^4(\mathbf{H}^n)^*$.

Remark 1.33. In the special case of $n = p = 1$, and $q = 0$, $\mathrm{HU}(1)$ acting on the left equals $\{L_a : |a| = 1\}$, while $\mathrm{HU}(1)$ acting on the right equals $\{R_b : |b| = 1\}$. In fact, the quaternionic unitary group is just the special orthogonal group. That is,

$$(1.34) \quad \mathrm{HU}(1) \cdot \mathrm{HU}(1) = \mathrm{SO}(4),$$

or equivalently,

$$(1.34') \quad \chi : \mathrm{HU}(1) \times \mathrm{HU}(1) \longrightarrow \mathrm{SO}(4),$$

is a surjective group homomorphism with kernel $\mathbf{Z}_2 = \{-1, 1\}$, where the map χ is defined by

$$\chi_{a,b}(x) \equiv ax\bar{b} \quad \text{for all } x \in \mathbf{H}.$$

To prove (1.34'), first note that by (1.29), or more directly, by Problem 6(b), $\det_{\mathbf{R}} L_a = 1$ if $|a| = 1$ (similarly $\det_{\mathbf{R}} R_b = 1$ if $|b| = 1$). Second, one can show that $|ax| = |a| |x|$ under quaternion multiplication. Thus, $L_a \in O(4)$ if $|a| = 1$ (similarly $R_b \in O(4)$ if $|b| = 1$). This proves that $\mathrm{HU}(1) \cdot \mathrm{HU}(1) \equiv \chi(\mathrm{HU}(1) \times \mathrm{HU}(1)) \subset \mathrm{SO}(4)$. The surjectivity of χ can be demonstrated with a topological argument based on dimension, once it is known that $\mathrm{SO}(4)$ is connected (see Corollary 3.31). A nontopological proof that χ is surjective is provided by Problem 4.9.

If $\mathrm{HU}(1)$ denotes the diagonal copy of $S^3 \equiv \{a \in \mathbf{H} : |a| = 1\}$ embedded in $\mathrm{HU}(1) \times \mathrm{HU}(1)$ and χ is restricted to $\mathrm{HU}(1)$, then

$$(1.35) \quad \frac{\mathrm{HU}(1)}{\mathbf{Z}_2} \stackrel{\chi}{\cong} \mathrm{SO}(3).$$

To prove (1.35), it suffices to note that the subgroup of $\mathrm{SO}(\mathbf{H})$ that fixes $1 \in \mathbf{H}$ is just $\mathrm{SO}(\mathrm{Im} \mathbf{H})$ and that the subgroup of $\mathrm{HU}(1) \times \mathrm{HU}(1)$ that maps into $\mathrm{SO}(\mathrm{Im} \mathbf{H})$ equals $\{(a, b) \in \mathrm{HU}(1) \times \mathrm{HU}(1) : a\bar{b} = 1\} = \mathrm{HU}(1)$.

The quaternionic enhancements are summarized as follows.

Group	Enhanced Group		
general linear	$GL(n, \mathbf{H})$	$GL(n, \mathbf{H}) \cdot \mathbf{H}^*$	enhanced general linear
special linear	$SL(n, \mathbf{H})$	$SL(n, \mathbf{H}) \cdot \mathbf{HU}(1)$	enhanced special linear
hyper-unitary	$\mathbf{HU}(p, q)$	$\mathbf{HU}(p, q) \cdot \mathbf{HU}(1)$	enhanced hyperunitary
skew-unitary	$\mathbf{SK}(n, \mathbf{H})$	$\mathbf{SK}(n, \mathbf{H}) \cdot \mathbf{HU}(1)$	enhanced skew unitary

Of course, one can always enhance a group G with \mathbf{R}^+ , or $\mathbf{R}^* \equiv \mathbf{R} - \{0\}$, if the nonzero multiples of the identity do not already belong to G . The groups $G \cdot \mathbf{R}^+$ are usually referred to as *conformal groups*. For example,

$$(1.36) \quad \mathbf{CO}(p, q) \equiv O(p, q) \cdot \mathbf{R}^+ \equiv O(p, q) \times \mathbf{R}^+$$

is called the *conformal (orthogonal) group of signature p, q* . This group, perhaps the most important conformal group, is usually defined by requiring that the inner product ε (see (1.10)) be fixed up to a positive scalar multiple (or conformal factor):

$$(1.36') \quad \begin{aligned} \mathbf{CO}(p, q) \equiv \{A \in GL(n, \mathbf{R}) : \text{for some } \lambda \in \mathbf{R}^+, \varepsilon(Ax, Ay) \\ = \lambda \varepsilon(x, y) \text{ for all } x, y \in \mathbf{R}^n\}. \end{aligned}$$

Similarly,

$$(1.37) \quad \begin{aligned} \mathbf{CSO}(p, q) \equiv \mathbf{SO}(p, q) \cdot \mathbf{R}^+ \\ \equiv \{A \in GL^+(n, \mathbf{R}) : A^* \varepsilon = \lambda \varepsilon \text{ for some } \lambda \in \mathbf{R}^+\} \end{aligned}$$

is called the *special conformal group of signature p, q* .

If both $p, q \geq 1$, then (see Chapter 3) $\mathbf{SO}(p, q)$ has two connected components. The connected component of the identity, denoted by $\mathbf{SO}^\uparrow(p, q)$, is, of course, a subgroup of $\mathbf{SO}(p, q)$. This subgroup $\mathbf{SO}^\uparrow(p, q)$ of $\mathbf{SO}(p, q)$ is called the *reduced special orthogonal group*. Later, in Chapter 4, additional subgroups of $O(p, q)$, denoted $O^+(p, q)$, and $O^-(p, q)$ will be discussed in some detail. Briefly, if $p, q \geq 1$, then $O(p, q)$ has four connected components. Adding any one of the remaining three components to $\mathbf{SO}^\uparrow(p, q)$ yields three additional subgroups of $O(p, q)$, denoted $\mathbf{SO}(p, q)$, $O^+(p, q)$, and $O^-(p, q)$. Thus, the intersection of any two of these three is always $\mathbf{SO}^\uparrow(p, q)$. See Chapter 4 for the details.

ISOMORPHISMS

The unit circle

$$(1.38) \quad S^1 \equiv \{z \in \mathbf{C} : |z| = 1\}$$

in the complex plane \mathbf{C} is a group under complex multiplication. By definition, the groups $U(1)$ and S^1 are the same. Of course, $S^1 \equiv \{e^{i\theta} : \theta \in \mathbf{R}\} \cong \mathbf{R}/2\pi\mathbf{Z}$. The group of nonzero complex numbers under complex multiplication is denoted by \mathbf{C}^* , and by definition, $GL(1, \mathbf{C}) \equiv \mathbf{C}^*$.

The set of unit quaternions

$$(1.39) \quad S^3 \equiv \{x \in \mathbf{H} : |x| = 1\}$$

also forms a group under quaternionic multiplication. Again, by definition, the groups $HU(1)$ and S^3 are the same.

Also, by definitions (1.20) and (1.20'), we have $SL(2, \mathbf{R}) = Sp(1, \mathbf{R})$ and $SL(2, \mathbf{C}) = Sp(1, \mathbf{C})$. The more difficult equality $HU(1) \cdot HU(1) = SO(4)$ has already been discussed. These and other coincidences are listed in the next proposition.

Proposition 1.40. *The following isomorphisms hold*

$$(1.41) \quad SO(2) \cong U(1) \cong SK(1) \cong S^1,$$

$$(1.42) \quad CSO(2) \cong GL(1, \mathbf{C}) \cong \mathbf{C}^* \cong SO(2, \mathbf{C}),$$

$$(1.43) \quad SU(2) \cong HU(1) \cong SL(1, \mathbf{H}) \cong S^3,$$

$$(1.44) \quad Sp(1, \mathbf{R}) = SL(2, \mathbf{R}) \cong SU(1, 1),$$

$$(1.45) \quad Sp(1, \mathbf{C}) = SL(2, \mathbf{C}),$$

$$(1.46) \quad HU(1) \cdot HU(1) \cong SO(4) \quad \text{and} \quad GL(1, \mathbf{H}) \cdot \mathbf{H}^* \cong CSO(4),$$

$$(1.47) \quad SO^\dagger(3, 1) \cong SO(3, \mathbf{C}).$$

The last isomorphism (1.47) will be verified in the section on special relativity in Chapter 3.

The proofs of all of the other isomorphisms in Proposition 1.40 are left as an exercise (see Problems 9, 10, and 11). One of these isomorphisms, $SU(2) \cong HU(1)$, warrants the following discussion.

Let \mathbf{H} have the complex structure $R_i x \equiv xi$ (right multiplication by i). Thus, $\mathbf{H} \cong \mathbf{C}^2$, where each $p \in \mathbf{H}$ can be expressed as $p = z + jw$ with

$z, w \in \mathbf{C} \subset \mathbf{H}$. Now each $A \in M_1(\mathbf{H}) \cong \text{End}_{\mathbf{H}}(\mathbf{H})$ can be considered as acting on \mathbf{H} on the left, hence $A \in \text{End}_{\mathbf{C}}(\mathbf{C}^2) \cong M_2(\mathbf{C})$ is a complex linear transformation of $\mathbf{H} \cong \mathbf{C}^2$. Using the coordinates $p = z + jw = (z, w)$ for $p \in \mathbf{H} \cong \mathbf{C}^2$, the complex linear map A expressed as a complex matrix is given by

$$A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix},$$

where $A = a + jb$, $a, b \in \mathbf{C} \subset \mathbf{H}$. This is because

$$Aj = (A \cdot 1)j = (a + jb)j = -\bar{b} + j\bar{a}.$$

This proves

$$(1.48) \quad M_1(\mathbf{H}) \cong \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) : a, b \in \mathbf{C} \right\}.$$

The isomorphism $\text{HU}(1) \cong \text{SU}(2)$ is derived from (1.48) (see Problem 10).

Remark. In the standard reference (Helgason [10]), $\text{SL}(n, \mathbf{H})$ is denoted by $\text{SU}^*(2n)$, $\text{SK}(n, \mathbf{H})$ is denoted by $\text{SO}^*(2n)$, and $\text{HU}(p, q)$ is denoted by $\text{Sp}(p, q)$.

SUMMARY

The three general linear groups $\text{GL}(n, \mathbf{R})$, $\text{GL}(n, \mathbf{C})$, and $\text{GL}(n, \mathbf{H})$ and the seven groups described in Table 1.18 can be changed by imposing restrictions on determinants and/or by enhancing with scalar multiplication. The connected component of the identity in $\text{SO}(p, q)$ with $p, q \geq 1$ is also a group. All the groups introduced in this chapter can be obtained in this way.

In low dimension, some of these groups coincide. One of the most interesting isomorphisms is $\text{SU}(2) \cong \text{HU}(1)$. The topic of special isomorphisms in low dimensions will be discussed again in Chapter 14.

PROBLEMS

1. Establish $M_n(F) \cong \text{End}_F(F^n)$ and $\text{GL}(n, F) \cong \text{GL}_F(F^n)$ for $F \equiv \mathbf{R}, \mathbf{C}, \mathbf{H}$.
2. If $A \in M_n(\mathbf{H}) \cong \text{End}_{\mathbf{H}}(\mathbf{H}^n)$ is injective, then A^{-1} is \mathbf{H} -linear.