References

[1] Bott, R. and Samelson, H. 'Applications of the theory of Morse to symmetric spaces', *American Journal of Mathematics* 80 (1958), 964-1029.

[2] Bott, R., 'An application of the Morse theory to the topology of Lie groups', *Bulletin de la Société Mathématique de France* 84 (1956), 251-282.

[3] Bott, R. and Samelson, H., 'The cohomology ring of G/T', *Proceedings of the National Academy of Sciences* 41 (1955), 490-493.

[4] Serre, J.-P., 'Représentations linéaires et espaces homegènes kähleriens des groupes de Lie compacts', Séminaire Bourbaki, No.100, Paris, 1954.

[5] Stiefel, E., 'Uber eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf die Aufzählung der einfachen Lie'schen Gruppen', *Commentarii Mathematici Helvetici* 14 (1941/2), 350-380.

[6] Stiefel, E., 'Kristellographishe Bestimmung der Characktere der geschlossenen Lieschen Gruppen, *Commentarii Mathematici Helvetici* 17 (1945), 165-200.

5 · Algebraic structure of Lie groups

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This survey of the algebraic structure of Lie groups and Lie algebras (mainly semisimple) is a considerably expanded version of the oral lectures at the symposium. It is limited to what is necessary for representation theory, which is another way of saying that very little has been left out. In spite of its length, it contains few proofs or even indications of proofs, nor have I given chapter and verse for each of the multitude of unproved assertions throughout the text. Instead, I have appended references to each section, from which the diligent reader should have no difficulty in tracking down the proofs.

I. Lie Groups and Lie Algebras

1. Vector fields

Let M be a smooth (C^∞) manifold, and for each point $x \in M$ let $T_x(M)$ denote the vector space of *tangent vectors* to M at x. The union of all the $T_x(M)$ is the *tangent bundle* $T(M)$ of M. Locally, if U if a coordinate neighbourhood in M, the restriction of $T(M)$ to U is just $U \times R^n$, where n is the dimension of M. Each smooth map $\phi : M \to N$, where N is another smooth manifold, gives rise to a *tangent map* $T(\phi) : T(M) \to T(N)$, whose restriction $T_x(\phi)$ to the tangent space $T_x(M)$ is a linear mapping of $T_x(M)$ into $T_{\phi(x)}(N)$. In terms of local coordinates in M and N, $T_x(\phi)$ is given by the Jacobian matrix. The familiar rule for differentiating a function of a function now takes the form $T(\phi \circ \psi) = T(\phi) \circ T(\psi)$, so that T is a functor (from smooth manifolds to smooth manifolds).

References

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I. Lie Groups and Lie Algebras

1. Vector fields

Let M be a smooth (C^∞) manifold, and for each point $x \in M$ let $T_x(M)$ denote the vector space of *tangent vectors* to M at x. The union of all the $T_x(M)$ is the *tangent bundle* $T(M)$ of M. Locally, if U if a coordinate neighbourhood in M, the restriction of $T(M)$ to U is just $U \times R^n$, where n is the dimension of M. Each smooth map $\phi : M \to N$, where N is another smooth manifold, gives rise to a *tangent map* $T(\phi) : T(M) \to T(N)$, whose restriction $T_x(\phi)$ to the tangent space $T_x(M)$ is a linear mapping of $T_x(M)$ into $T_{\phi(x)}(N)$. In terms of local coordinates in M and N, $T_x(\phi)$ is given by the Jacobian matrix. The familiar rule for differentiating a function of a function now takes the form $T(\phi \circ \psi) = T(\phi) \circ T(\psi)$, so that T is a functor (from smooth manifolds to smooth manifolds).

In particular, if $N = \mathbb{R}$, each tangent space $T_y(N)$ may be canonically identified with $\mathbb{R}$. Hence if f is a smooth real-valued function defined on an open neighbourhood of $x \in M$, and ξ is a tangent vector at x, then $T_x(f).\xi$ is a real number, the *directional derivative* of f at x in the direction ξ.

A (smooth) *vector field* on M is a function X which assigns to each $x \in M$ a tangent vector $X(x) \in T_x(M)$, varying smoothly with x; in other words, X is a smooth section of the tangent bundle $T(M)$. X acts on smooth functions as follows:

$$(Xf)(x) = T_x(f).X(x) \quad .$$

In this way X acts as a *derivation* of the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth functions on M; that is to say, X is $\mathbb{R}$-linear and satisfies

$$X(fg) = (Xf).g + f.Xg \tag{1}$$

for $f,g \in C^\infty(M)$; this is just the expression, in the present context, of the rule for differentiating a product of two functions. Conversely, each derivation of $C^\infty(M)$ arises in this way from a unique vector field, and we may therefore *identify* X with the derivation it defines.

Now let X and Y be vector fields (or derivations) on M. Then $X \circ Y : C^\infty(M) \to C^\infty(M)$ is not a derivation, but the Lie bracket

$$[X,Y] = X \circ Y - Y \circ X$$

always is (just check that (1) is satisfied). It follows that the space of vector fields on a manifold M has the structure of a *Lie algebra* over $\mathbb{R}$: it is a (usually infinite-dimensional) vector space over $\mathbb{R}$, equipped with a 'Lie bracket' $[X,Y]$ which is $\mathbb{R}$-bilinear and anticommutative, and in addition satisfies the 'Jacobi identity'

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 .$$

2. The Lie algebra of a Lie group

A *Lie group* G is a smooth manifold which is also a group, the two structures being compatible: that is to say, the mappings $m : G \times G \to G$ and $i : G \to G$ defined by multiplication and inversion $(m(x,y) = xy,\ i(x) = x^{-1})$ are *smooth*.

Examples

1. Any discrete group may be regarded as a Lie group (of dimension 0).
2. The additive group of $\mathbb{R}^n$ (or of any finite-dimensional real vector space) is a Lie group. Such a group is called a *vector group*.
3. The *circle group* $T = \mathbb{R}/\mathbb{Z}$ is a Lie group. The *n-dimensional torus* $T^n = (\mathbb{R}/\mathbb{Z})^n$ is a Lie group.
4. The general linear group $GL(n,\mathbb{R})$ of invertible real $n \times n$ matrices is an open submanifold of the space $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n \times n$ matrices, since it is the complement of the hypersurface $\det X = 0$. Hence $GL(n,\mathbb{R})$ is a Lie group, of dimension n^2. It is not connected but has two components, corresponding to positive and negative determinant. The identity component, consisting of the matrices X with $\det X > 0$, is denoted by $GL^+(n,\mathbb{R})$.

   More intrinsically, if V is a real vector space of dimension n, the group $GL(V)$ of invertible linear transformations of V is a Lie group, isomorphic to $GL(n,\mathbb{R})$.
5. Likewise $GL(n,\mathbb{C})$, the group of invertible complex $n \times n$ matrices, is a (complex) Lie group, of complex dimension n^2. Unlike $GL(n,\mathbb{R})$, it is connected.
6. Let $\mathbb{H}$ denote the division ring of quaternions. Then

$GL(n,H)$ is a (real) Lie group of dimension $4n^2$.

For each $x \in G$, let $\lambda_x : G \to G$ denote left translation by x:

$$\lambda_x(y) = xy \quad .$$

Clearly λ_x is a diffeomorphism of G, its inverse being $\lambda_{x^{-1}}$.

Let X be a vector field on G. We say that X is *left-invariant* if X commutes with left translations, i.e. if

$$X \circ \lambda_x = T(\lambda_x) \circ X$$

for all $x \in G$. If we regard X as a derivation, left-invariance is expressed by

$$(Xf) \circ \lambda_x = X(f \circ \lambda_x)$$

for all $f \in C^\infty(G)$ and $x \in G$. It follows immediately that the space of left-invariant vector fields on G is closed under the Lie bracket, and is therefore a *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$, called the *Lie algebra* of the Lie group G.

Each $X \in \mathfrak{g}$ is determined by its value $X(e) \in T_e(G)$ at the identity element e of G, because

$$X(x) = (X \circ \lambda_x)(e) = T(\lambda_x)X(e) \quad .$$

Conversely, each tangent vector $\xi \in T_e(G)$ determines a left-invariant vector field X_ξ on G by the rule

$$X_\xi(x) = T_e(\lambda_x)\xi \quad .$$

Consequently $\mathfrak{g}$ may be identified with $T_e(G)$, the tangent space to G at the identity element e. In particular it follows that $\dim \mathfrak{g} = \dim G$.

We may also remark here that the tangent bundle $T(G)$ of a Lie group G is *trivial*, i.e. is isomorphic (as a bundle) to $G \times T_e(G)$. Indeed, the mapping $(x,\xi) \to X_\xi(x)$ is an isomorphism of $G \times T_e(G)$ onto $T(G)$.

Now let H be another Lie group and let $\phi : G \to H$ be a smooth homomorphism; let $\mathfrak{g} = T_e(G)$, $\mathfrak{h} = T_e(H)$. The tangent map $T_e(\phi) : \mathfrak{g} \to \mathfrak{h}$ is called the *derived homomorphism* of ϕ and is denoted by ϕ_*. It is a homomorphism of Lie algebras, i.e. we have $\phi_*[X,Y] = [\phi_*X, \phi_*Y]$ for $X,Y \in \mathfrak{g}$. •

Examples

1. If $G = R^n$, then $\mathfrak{g} = R^n$ and $[X,Y] = 0$ for all $X,Y \in \mathfrak{g}$. For each $X \in \mathfrak{g}$, regarded as a derivation of $C^\infty(R^n)$, is of the form $X = \sum_{i=1}^{n} a_i \partial/\partial x_i$, with *constant* coefficients a_i; any two such derivations clearly commute, because $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$ on smooth functions.
2. Let $G = GL(n,R)$. Define $\alpha : \mathfrak{g} \to M(n,R)$ by

$$\alpha(X)_{ij} = (Xx_{ij})(I_n) \qquad (1 \le i,j \le n)$$

where I_n is the unit matrix (the identity element of G) and $x_{ij} : G \to R$ assigns to each matrix in G its (i,j) element. Then α is an isomorphism of vector spaces and $\alpha[X,Y] = \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X)$. The Lie algebra $\mathfrak{gl}(n,R)$ of $GL(n,R)$ is therefore canonically identified with the Lie algebra of all $n \times n$ matrices. Likewise with C or H in place of R.
3. If V is a real vector space of dimension n (so that $V \cong R^n$), the Lie algebra of $GL(V)$ ($\cong GL(n,R)$) is denoted by $\mathfrak{gl}(V)$. As in Ex.2 we may identify $\mathfrak{gl}(V)$ with the Lie algebra of the ring $\mathrm{End}(V)$ of all linear transformations of V.
4. If G is an abelian Lie group, then $\mathfrak{g}$ is an abelian Lie algebra, i.e. $[X,Y] = 0$ for all $X,Y \in \mathfrak{g}$.

Remark. We could of course have started with *right*-invariant vector fields. However, the inversion $i : x \to x^{-1}$ interchanges right and left, and we get nothing new.

3. The exponential map

The usual exponential function e^x is a smooth mapping from $R = \mathfrak{gl}(1,R)$ onto $GL^+(1,R)$, the multiplicative group of positive real numbers. More generally, if X is any real $n \times n$ matrix, the exponential series $\sum_{n=0}^{\infty} X^n/n!$ converges in the space $M(n,R)$ of $n \times n$ matrices, and its sum $e^X = \exp(X)$ is invertible (with inverse e^{-X}) and has positive determinant (namely $e^{\text{trace } X}$). Hence $X \to e^X$ is a smooth function on $M(n,R) = \mathfrak{gl}(n,R)$ with values in $GL^+(n,R)$. These examples are particular instances of the *exponential map*, which is defined for any Lie group G, and is a smooth mapping of the Lie algebra $\mathfrak{g}$ into the group G.

The definition runs as follows. A *one-parameter subgroup* of G is a smooth homomorphism $u : R \to G$. Its derived homomorphism $u_* = T_0(u)$ is a linear mapping of R into $\mathfrak{g}$, the Lie algebra of G. It is a consequence of the theorem of existence and uniqueness of solutions of linear ordinary differential equations that the mapping $u \to u_*(1)$ is a bijection of the set of one-parameter subgroups of G onto the Lie algebra $\mathfrak{g}$: for each $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup $u_X : R \to G$ such that $u_{X*}(1) = X$. The exponential map $\exp_G : \mathfrak{g} \to G$ is now defined by

$$\exp_G(X) = u_X(1) \quad .$$

We have $\exp(tX) = u_X(t)$ for all $t \in R$, so that $\exp(sX)\exp(tX) = \exp((s+t)X)$.

The exponential map is a smooth map whose derivative at $0 \in \mathfrak{g}$ is $1_{\mathfrak{g}}$, the identity mapping of $\mathfrak{g}$. Hence, by the inverse function theorem, $\exp$ is a diffeomorphism of some open neighbourhood of 0 in $\mathfrak{g}$ onto an open neighbourhood of e in G ; that is to say, it provides a chart of G around the identity element. From this it follows that, if G is connected, the image $\exp(\mathfrak{g})$ of the exponential map *generates* G (although in general $\exp : \mathfrak{g} \to G$ is not surjective, except in the cases where G is compact or abelian (and connected)).

For $X, Y \in \mathfrak{g}$ and $t \in R$ we have

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2)) \quad , \tag{1}$$

$$[\exp(tX), \exp(tY)] = \exp(t^2[X,Y] + O(t^3)) \tag{2}$$

(where on the left-hand side of (2) the bracket is the commutator $[x,y] = xyx^{-1}y^{-1}$ in G). Thus, under the exponential map, multiplication in G corresponds approximately to addition in $\mathfrak{g}$, and commutator formation in G corresponds approximately to the Lie bracket in $\mathfrak{g}$.

If G is abelian, $\exp_G$ is additive, and therefore a homomorphism of the vector group $\mathfrak{g}$ into G.

If $\phi : G \to H$ is a smooth homomorphism, then we have

$$\phi \circ \exp_G = \exp_H \circ \phi_*$$

(*naturality* of $\exp$).

4. The adjoint representation

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. For each $x \in G$, let $\text{Int}(x) : g \to xgx^{-1}$ be the inner automorphism of G defined by x. $\text{Int}(x)$ is a smooth automorphism of G, and its derived homomorphism is denoted by $\text{Ad}(x)$ or $\text{Ad}_G(x)$:

$$\text{Ad}(x) = \text{Int}(x)_* : \mathfrak{g} \to \mathfrak{g}$$

is an automorphism of the Lie algebra $\mathfrak{g}$, *a fortiori* of the vector space $\mathfrak{g}$. Since $\mathrm{Int}(x) \circ \mathrm{Int}(y) = \mathrm{Int}(xy)$, we have $\mathrm{Ad}(x) \circ \mathrm{Ad}(y) = \mathrm{Ad}(xy)$; also $\mathrm{Ad}(x)$ varies smoothly with x , and therefore

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

is a smooth homomorphism of G into the general linear group of $\mathfrak{g}$, called the *adjoint representation* of G .

If G is connected, the kernel of Ad is the centre of G .

The derived homomorphism of Ad is denoted by $\mathrm{ad}_{\mathfrak{g}}$ or ad :

$$\mathrm{Ad}_* = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$, called the *adjoint representation* of $\mathfrak{g}$. More directly (and without reference to G), ad $\mathfrak{g}$ may be defined by

$$(\mathrm{ad}\, X)Y = [X,Y]$$

for $X,Y \in \mathfrak{g}$. That $[\mathrm{ad}\, X, \mathrm{ad}\, Y] = \mathrm{ad}[X,Y]$ is just a restatement of the Jacobi identity (§1).

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The *normalizer* $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G is the group of all $x \in G$ such that $\mathrm{Ad}(x)\mathfrak{h} \subset \mathfrak{h}$, and the *centralizer* $Z_G(\mathfrak{h})$ of $\mathfrak{h}$ in G is the group of all $x \in G$ such that $\mathrm{Ad}(x)|\mathfrak{h} = 1_{\mathfrak{h}}$. Likewise, the *normalizer* $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the subalgebra of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)\mathfrak{h} \subset \mathfrak{h}$, and the *centralizer* $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the subalgebra of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)|\mathfrak{h} = 0$. $N_G(\mathfrak{h})$ and $Z_G(\mathfrak{h})$ are closed subgroups of G and hence (§5) are Lie groups. The Lie algebra of $N_G(\mathfrak{h})$ (resp. $Z_G(\mathfrak{h})$) is $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})$ (resp. $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{h})$).

Let $\mathfrak{g}$ be a (finite-dimensional) real Lie algebra, and consider the polynomial in t

$$\det(t - \mathrm{ad}_{\mathfrak{g}}(X)) = \sum_{i=0}^{n} d_i(X)t^i \qquad (X \in \mathfrak{g})$$

of degree $n = \dim \mathfrak{g}$. The d_i are polynomial functions in $\mathfrak{g}$. The smallest integer ℓ such that $d_\ell \neq 0$ is called the *rank* of $\mathfrak{g}$, and an element $X \in \mathfrak{g}$ is said to be *regular* if $d_\ell(X) \neq 0$. The set $\mathfrak{g}'$ of regular elements in $\mathfrak{g}$ is therefore the complement of a real algebraic variety in $\mathfrak{g}$, and hence is a dense open subset of $\mathfrak{g}$.

These definitions have global counterparts. Let G be a connected Lie group, and consider the polynomial in t

$$\det(t + 1 - \mathrm{Ad}_G(x)) = \sum_{i=0}^{n} D_i(x)t^i \qquad (x \in G).$$

of degree $n = \dim G$. The D_i are real analytic functions on G . The least integer ℓ such that $D_\ell \neq 0$ is called the *rank* of G , and an element $x \in G$ is said to be *regular* if $D_\ell(x) \neq 0$. We have $\mathrm{rank}(G) = \mathrm{rank}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G . The set G' of regular elements of G is a dense open subset of G , stable under inner automorphisms, whose complement has measure zero with respect to (left or right) Haar measure on G .

5. Subgroups and subalgebras

By a *Lie subgroup* of a Lie group G we mean a (locally closed) submanifold H of G which is also a subgroup of G . It is almost immediate that H is a closed subgroup of G and a Lie group. The converse of this result is also true, but harder to prove: *every closed subgroup* H *of a Lie group* G *is a submanifold of* G (and therefore a Lie subgroup of G) (E. Cartan's theorem). The Lie algebra $\mathfrak{h}$ of H consists of all $X \in \mathfrak{g}$ such that $\exp(tX) \in H$ for all $t \in \mathbb{R}$.

Examples

1. The *special linear group* $SL(n,\mathbb{R})$, consisting of the real $n \times n$ matrices with determinant 1 , is closed in $GL(n,\mathbb{R})$, hence is a Lie group. Its Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ consists of

the $X \in \mathfrak{gl}(n,R)$ with trace $X = 0$ (because $\det(e^X) = e^{\operatorname{trace} X}$). Likewise with C in place of R.

2. Let K be any one of R, C or H. Let $\bar{x}$ denote the conjugate of $x \in K$ (so that $\bar{x} = x$ if $K = R$). Let $U(n,K)$ denote the group of all $X \in GL(n,K)$ such that $X\bar{X}^t = 1$. Then $U(n,K)$ is a subgroup of $GL(n,K)$, and is *closed* because it is defined by the polynomial equations

$$\sum_{k=1}^{n} x_{ik}\bar{x}_{jk} = \delta_{ij} ,$$

hence by Cartan's theorem is a Lie group. These equations also imply that $\sum_{i,j} |x_{ij}|^2 = n$, so that $U(n,K)$ is a *bounded* subset of $M(n,K)$, and is therefore *compact*. Hence $U(n,K)$ is a compact Lie group, and its Lie algebra consists of all $X \in M(n,K)$ such that $X + \bar{X}^t = 0$, i.e. such that X is skew-Hermitian (or skew-symmetric, when $K = R$).

(i) When $K = R$, $U(n,K)$ is the *orthogonal group* $O(n)$, which has two components (corresponding to determinant $+1$ and -1). The *special orthogonal group* $SO(n)$, consisting of the orthogonal matrices with determinant $+1$, is a compact connected Lie group. Its Lie algebra $\mathfrak{so}(n)$ consists of the real skew-symmetric $n \times n$ matrices of trace 0.

(ii) When $K = C$, $U(n,K)$ is the *unitary group* $U(n)$, which is connected. The *special unitary group* $SU(n)$, consisting of the unitary matrices $X \in U(n)$ with $\det X = 1$, is a closed subgroup of $U(n)$ and therefore also a Lie group. Its Lie algebra $\mathfrak{su}(n)$ consists of the complex skew-Hermitian $n \times n$ matrices with trace 0.

(iii) When $K = H$, $U(n,K)$ is the *quaternionic unitary group* $Sp(n)$.

If H is a Lie subgroup of G, the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G. Conversely, however an arbitrary Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *not* necessarily the Lie algebra of a Lie subgroup of G. What *is* true

is that to each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exists a connected Lie group H and a smooth injective homomorphism $j : H \to G$ such that j_* is an isomorphism of the Lie algebra of H onto $\mathfrak{h}$; and the pair (H,j) is unique up to isomorphism. The image $j(H)$ is the subgroup of G generated by $\exp_G(\mathfrak{h})$. The connected Lie group H, identified with its image in G, is called the *immersed* subgroup of G corresponding to $\mathfrak{h}$; in general it is not closed in G, and the topology of the Lie group H is not the topology induced from G.

Example. Let G be the torus T^2, so that $\mathfrak{g} = R^2$; let $\mathfrak{h} = R$, embedded in R^2 by $x \to (x,\theta x)$ where θ is an irrational number. Then $H = R$, and $j(H) \subset G$ is a curve which winds round and round the torus infinitely often, so that $j(H)$ is *dense* in G.

The correspondence between subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ and immersed subgroups H of G has all the properties that one could reasonably expect. The centralizer (resp. normalizer) of H in G is equal to the centralizer (resp. normalizer) of $\mathfrak{h}$ in G. In particular, if G is connected, H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (i.e. $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$), and the centre C of G has Lie algebra $\mathfrak{c}$, the centre of $\mathfrak{g}$ (i.e. $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{g})$). Again, if G is connected, the derived group DG (generated by all commutators $[x,y]$) is an immersed subgroup which corresponds to the derived algebra $\mathfrak{D}\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$, spanned by all brackets $[X,Y]$. It follows that a connected Lie group is solvable if and only if its Lie algebra is solvable (i.e. $\mathfrak{D}^r\mathfrak{g} = 0$ for some $r \geq 1$).

Example. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\operatorname{Aut}(\mathfrak{g})$ its group of automorphisms. $\operatorname{Aut}(\mathfrak{g})$ is a closed (indeed algebraic) subgroup of $GL(\mathfrak{g})$, hence is a Lie group. Its Lie algebra is the algebra $\operatorname{Der}(\mathfrak{g})$ of *derivations* of $\mathfrak{g}$, a subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

The image $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{g}$ under the adjoint representation (§4) is a subalgebra of $\mathrm{Der}(\mathfrak{g})$. To it there corresponds an immersed subgroup $\mathrm{Int}(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$, called the *adjoint group* of $\mathfrak{g}$; it is generated by the automorphisms $\exp(\mathrm{ad}\, X)$, $X \in \mathfrak{g}$. If G is a connected Lie group with $\mathfrak{g}$ as Lie algebra, then $\mathrm{Int}(\mathfrak{g})$ is the image of G under the adjoint representation (because $\exp(\mathrm{ad}\, X) = \mathrm{Ad}(\exp X)$ by naturality of $\exp$).

If $\mathfrak{g}$ is semisimple, $\mathrm{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$, and $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ (i.e., every derivation of $\mathfrak{g}$ is inner).

6. Quotients

Let G be a Lie group, H a *closed* subgroup of G . The quotient set G/H , whose elements are the cosets $xH = \dot{x}$ of H in G , then carries a unique structure of a smooth manifold such that the projection $p : x \to \dot{x}$ of G onto G/H is smooth, and such that a mapping f of G/H into a smooth manifold M is smooth if and only if $f \circ p : G \to M$ is smooth. The tangent space to G/H at the image $\dot{e}$ of e is $\mathfrak{g}/\mathfrak{h} = T_e(G)/T_e(H)$, from which it follows that $\dim(G/H) = \dim G - \dim H$. Moreover, the projection p has a smooth local cross-section defined on an open neighbourhood of $\dot{e}$, from which it follows that locally G looks like the Cartesian product of H with G/H , or more precisely that G is a smooth bundle over G/H with fibre H .

If H is a closed *normal* subgroup of G , then the group structure and the manifold structure on G/H are compatible, i.e. G/H is a Lie group.

Example. If G is a connected Lie group, then $\mathrm{Ad}(G) \cong G/Z$ where Z is the centre of G .

7. Homomorphisms and local homomorphisms

Let $\phi : G \to H$ be a smooth homomorphism of Lie groups. The *kernel* N of ϕ is closed in G , hence is a Lie subgroup of G , whose Lie algebra is the kernel of the derived homomorphism $\phi_* : \mathfrak{g} \to \mathfrak{h}$. The *image* $\phi(G)$, on the other hand, need not be a closed subgroup of H but (provided that G is connected) is the *immersed* subgroup of H corresponding to the subalgebra $\phi_*(\mathfrak{g})$ of $\mathfrak{h}$. The immersion is the injective smooth homomorphism $G/N \to H$ induced by ϕ .

For example, the one-parameter subgroups of G (§3) are immersed subgroups.

Let G and H again be Lie groups. A (smooth) *local homomorphism* from G to H is a smooth mapping ϕ of an open neighbourhood U of the identity element in G , with values in H , such that $\phi(xy) = \phi(x)\phi(y)$ whenever x, y and xy all lie in U . If ϕ is also a diffeomorphism of U onto an open neighbourhood of the identity element in H , then ϕ^{-1} is a local homomorphism from H to G , and ϕ is said to be a *local isomorphism* of G with H .

Each local homomorphism ϕ from G to H has a *derived homomorphism* $\phi_* = T_e(\phi) : \mathfrak{g} \to \mathfrak{h}$, which is a homomorphism of Lie algebras; and ϕ is a local isomorphism if and only if ϕ_* is an isomorphism.

Conversely, if $u : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, there exists a local homomorphism ϕ from G to H such that $u = \phi_*$, and moreover ϕ is essentially unique (in the sense that if $u = \phi_{1*} = \phi_{2*}$, then ϕ_1 and ϕ_2 coincide on some open neighbourhood of e in G). It follows that two Lie groups G,H are locally isomorphic if and only if their Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ are isomorphic.

If now G is connected and simply-connected, every local homomorphism from G to H has a unique extension to a (global) smooth homomorphism of G into H (monodromy theorem). Hence the smooth homomorphisms of a connected and simply-connected Lie group G into any Lie group H are in one-one correspondence (via the derived homomorphism) with the Lie algebra homomorphisms of $\mathfrak{g}$ into $\mathfrak{h}$.

8. The universal covering group

Let G be a connected Lie group. Then G has a *universal covering* group $\tilde{G}$, which is a Lie group, characterized up to isomorphism by the following properties: (i) there exists a surjective smooth homomorphism $p : \tilde{G} \to G$ with discrete kernel; (ii) $\tilde{G}$ is connected and simply-connected. The kernel D of p is isomorphic to the fundamental group $\pi_1(G)$, and is a subgroup of the centre of $\tilde{G}$ (because for each $d \in D$ the mapping $x \to xdx^{-1}$ of $\tilde{G}$ into D is continuous, and therefore constant). Hence D , and therefore also $\pi_1(G)$, is *abelian*.

The derived homomorphism $p_* : \mathrm{Lie}(\tilde{G}) \to \mathrm{Lie}(G) = \mathfrak{g}$ is an isomorphism. Hence the connected Lie groups with $\mathfrak{g}$ as Lie algebra are all obtained from $\tilde{G}$ by factoring out a discrete subgroup of the centre of $\tilde{G}$.

Finally, every (finite-dimensional) Lie algebra $\mathfrak{g}$ is the Lie algebra of some connected Lie group G , hence also of its universal covering $\tilde{G}$. In this way is established a one-one correspondence between isomorphism classes of finite-dimensional real Lie algebras and isomorphism classes of connected and simply-connected Lie groups. Thus, for an arbitrary Lie group G , the only information about G that is not captured by its Lie algebra $\mathfrak{g}$ is (i) properties that depend on the different connected components, (ii) properties which depend on different covering groups, i.e. on $\pi_1(G)$.

II. Semisimple Lie Algebras

1. Generalities on Lie algebras

Many of the notions of group theory have counterparts for Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (over any field of characteristic 0). If $\mathfrak{a}$, $\mathfrak{b}$ are vector subspaces of $\mathfrak{g}$, we denote by $[\mathfrak{a},\mathfrak{b}]$ the vector space spanned by all $[X,Y]$ with $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. A vector subspace $\mathfrak{a}$ of $\mathfrak{g}$ is a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a}$, and an *ideal* in $\mathfrak{g}$ if $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{a}$: these are the counterparts of the notions of subgroup and normal subgroup, respectively. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ we can (as in other algebraic contexts) factor it out to form the quotient algebra $\mathfrak{g}/\mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$, then $[\mathfrak{a},\mathfrak{b}]$ is also an ideal.

The *derived series* of $\mathfrak{g}$ is the decreasing sequence of ideals $(\mathfrak{D}^r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{D}^0\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{D}^{r+1}\mathfrak{g} = [\mathfrak{D}^r\mathfrak{g},\mathfrak{D}^r\mathfrak{g}]$. Just as in group theory, if $\mathfrak{D}^r\mathfrak{g} = 0$ for some r , the Lie algebra $\mathfrak{g}$ is said to be *solvable*.

The *lower central series* of $\mathfrak{g}$ is the decreasing sequence of ideals $(\mathfrak{C}^r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{C}^0\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{C}^{r+1}\mathfrak{g} = [\mathfrak{g},\mathfrak{C}^r\mathfrak{g}]$. The *upper central series* of $\mathfrak{g}$ is the increasing sequence of ideals $(\mathfrak{C}_r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{C}_0\mathfrak{g} = 0$ and $\mathfrak{C}_{r+1}\mathfrak{g}/\mathfrak{C}_r\mathfrak{g}$ is the centre of $\mathfrak{g}/\mathfrak{C}_r\mathfrak{g}$. Just as in group theory, we have $\mathfrak{C}^r\mathfrak{g} = 0$ for large r if and only if $\mathfrak{C}_r\mathfrak{g} = 0$ for large r , and the Lie algebra $\mathfrak{g}$ is then said to be *nilpotent*. An equivalent condition is that $\mathrm{ad}_{\mathfrak{g}}X$ should be nilpotent for all $X \in \mathfrak{g}$.

Every nilpotent Lie algebra is solvable, and a Lie algebra $\mathfrak{g}$ is solvable if and only if its derived algebra $\mathfrak{D}\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ is nilpotent.

The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $B_{\mathfrak{g}}$ defined by

$$B_{\mathfrak{g}}(X,Y) = \mathrm{trace}(\mathrm{ad}\ X)(\mathrm{ad}\ Y)$$

If $\mathfrak{g}$ is nilpotent, $B_{\mathfrak{g}}$ is identically zero; conversely, if $B_{\mathfrak{g}} = 0$ then $\mathfrak{g}$ is solvable.

Let $\mathfrak{g}$ again be any finite dimensional Lie algebra. Then $\mathfrak{g}$ has a unique maximal solvable ideal (for if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals, then so is $\mathfrak{a}+\mathfrak{b}$). This ideal $\mathfrak{r}$ is called the *radical* of $\mathfrak{g}$. It is also the orthogonal complement of the derived algebra $\mathfrak{D}\mathfrak{g}$ with respect to the Killing form.

If $\mathfrak{r} = 0$, that is if $\mathfrak{g}$ has no nonzero solvable ideals, then $\mathfrak{g}$ is said to be *semisimple*. An equivalent condition is that $\mathfrak{g}$ is a direct product of *simple* Lie algebras (a Lie algebra is *simple* if it has no nontrivial ideals *and is not abelian*). Yet another equivalent condition is that the Killing form $B_{\mathfrak{g}}$ is nondegenerate.

If $\mathfrak{g}$ is again any finite-dimensional Lie algebra, $\mathfrak{r}$ its radical, then there exists a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{r} \qquad (1)$$

(direct sum). $\mathfrak{l}$ is called a *Levi subalgebra* of $\mathfrak{g}$, and (1) is a *Levi decomposition*. The algebra $\mathfrak{l}$ is semisimple, because it is isomorphic to $\mathfrak{g}/\mathfrak{r}$, which has zero radical. The algebra $\mathfrak{l}$ in (1) is not uniquely determined, but any two are conjugate in $\mathfrak{g}$ under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (I, §5). Also, the Levi subalgebras of $\mathfrak{g}$ are the maximal semisimple subalgebras of $\mathfrak{g}$.

If the radical $\mathfrak{r}$ is the centre $\mathfrak{z}$ of $\mathfrak{g}$, the Lie algebra $\mathfrak{g}$ is said to be *reductive*. An equivalent condition is that the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ should be completely reducible. If $\mathfrak{g}$ is reductive, its derived algebra $\mathfrak{D}\mathfrak{g}$ is semisimple, and $\mathfrak{g}$ is the direct product of $\mathfrak{D}\mathfrak{g}$ and $\mathfrak{z}$. Hence the reductive Lie algebras are just direct products of abelian and semisimple Lie algebras, and we shall therefore concentrate on the latter.

Examples. $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{u}(n)$ are reductive but not semisimple. $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ are semisimple (e.g. by computing the Killing form explicitly).

In particular, if $\mathfrak{g}$ is semisimple, the centre of $\mathfrak{g}$ is zero and hence the adjoint representation embeds $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$, the Lie algebra of $GL(\mathfrak{g})$. From the results of Chap. I, §5 it follows that $\mathfrak{g}$ is isomorphic to the Lie algebra of an immersed subgroup G of $GL(\mathfrak{g})$. Hence every semisimple real Lie algebra is the Lie algebra of some connected Lie group. One can then use Levi's theorem above to show that *every* finite-dimensional Lie algebra over $\mathbb{R}$ is the Lie algebra of a connected Lie group.

If $\mathfrak{g}$ is a real Lie algebra, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ its complexification, then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semisimple. For the matrix of the Killing form, relative to a basis of $\mathfrak{g}$, is the same for $\mathfrak{g}_{\mathbb{C}}$ as for $\mathfrak{g}$. If $\mathfrak{g}$ is *simple*, then $\mathfrak{g}_{\mathbb{C}}$ is either simple or is the product of two isomorphic simple algebras.

If $\mathfrak{g}$ is a *complex* Lie algebra, let $\mathfrak{g}^{\mathbb{R}}$ denote $\mathfrak{g}$ regarded as a real Lie algebra. If $\mathfrak{g}$ is semisimple (resp. simple) then so is $\mathfrak{g}^{\mathbb{R}}$. We have $(\mathfrak{g}^{\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$. We call $\mathfrak{g}^{\mathbb{R}}$ the *realification* of $\mathfrak{g}$.

A subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}^{\mathbb{R}}$ is a *real form* of the complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$. The real simple Lie algebras are either real forms or realifications of complex simple Lie algebras. We shall begin with the structure theory of the complex Lie algebras.

2. Cartan subalgebras

Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. An element $X \in \mathfrak{g}$ is *semisimple* if the linear transformation $\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$ is semisimple (i.e. diagonalizable over $\mathbb{C}$). A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ consisting of semisimple elements; equivalently, it is the centralizer in $\mathfrak{g}$ of a *regular* element of $\mathfrak{g}$ (Chapter I, §4).

Now let $\mathfrak{g}$ be *complex*. The importance of the Cartan subalgebras for unravelling the structure of $\mathfrak{g}$ lies in the fundamental fact that they are all conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (Chapter I, §5). (As we shall see later, this is not in general true for real semisimple Lie algebras, and is one of the reasons why their structure theory is more complicated.)

The (complex) dimension of a Cartan subalgebra of $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$, as defined in I, §4. We shall denote it by ℓ.

Example. If $\mathfrak{g} = \mathfrak{sl}(n,C)$, the diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra. Hence the rank of $\mathfrak{sl}(n,C)$ is $n-1$.

3. Roots

Until further notice, $\mathfrak{g}$ is a complex semisimple Lie algebra. Since all the Cartan subalgebras of $\mathfrak{g}$ are conjugate, there is no harm in choosing one, say $\mathfrak{h}$, once and for all. Since $\mathfrak{h}$ is abelian and the ground field C is algebraically closed, the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, restricted to $\mathfrak{h}$, splits up as a direct sum of one-dimensional representations. In other words, if $\mathfrak{h}^*$ is the vector space dual of $\mathfrak{h}$, and if for each $\alpha \in \mathfrak{h}^*$ we denote by $\mathfrak{g}^{\alpha}$ the subspace of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(H).X = \alpha(H)X$ for all $H \in \mathfrak{h}$, then $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}^{\alpha}$. Two such subspaces $\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}$ are orthogonal with respect to the Killing form B unless $\alpha+\beta = 0$. Moreover, $\mathfrak{g}^{0}$ is equal to $\mathfrak{h}$, because $\mathfrak{h}$ is its own centralizer in $\mathfrak{g}$. It follows that $\mathfrak{h}$ is orthogonal to all the $\mathfrak{g}^{\alpha}$, $\alpha \neq 0$, and therefore the restriction of B to $\mathfrak{h}$ remains nondegenerate.

Since $\mathfrak{g}$ is finite-dimensional, only finitely many of the $\mathfrak{g}^{\alpha}$ are nonzero. If $\alpha \neq 0$ and $\mathfrak{g}^{\alpha} \neq 0$, then α is said to be a *root* of $\mathfrak{g}$ (relative to the Cartan subalgebra $\mathfrak{h}$) and

$\mathfrak{g}^{\alpha}$ the *root-space* of α. If α is a root, so is $-\alpha$ (otherwise $\mathfrak{g}^{\alpha}$ would be orthogonal to all of $\mathfrak{g}$, contrary to the nondegeneracy of B). For each root α, we have $\dim \mathfrak{g}^{\alpha} = 1$.

If H is a general element of $\mathfrak{h}$, the complex numbers $\alpha(H)$ (α a root) are the nonzero eigenvalues of the linear transformation $\mathrm{ad}_{\mathfrak{g}}H$, i.e. they are the nonzero *roots* of the characteristic equation $\det(\lambda - \mathrm{ad}\,H) = 0$; this is the reason for the terminology.

We denote by R or $R(\mathfrak{g},\mathfrak{h})$ the set of roots: it is a finite subset of $\mathfrak{h}^*$. We have then a direct decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}^{\alpha} \tag{1}$$

called the *root-space* decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$.

The roots span a real subspace V of dimension ℓ in $\mathfrak{h}^*$, so that $\mathfrak{h}^*$ is the complexification of V. We have already observed that the Killing form B remains nondegenerate in restriction to $\mathfrak{h}$, hence defines an isomorphism $\lambda \to H_{\lambda}$ of $\mathfrak{h}^*$ onto $\mathfrak{h}$, and a bilinear form $\langle\lambda,\mu\rangle = B(H_{\lambda},H_{\mu})$ on $\mathfrak{h}^*$. It turns out that the restriction of this to V is real-valued and positive-definite, so that V acquires the structure of a real Euclidean space. Let $\mathfrak{h}_R$ denote the vector space spanned by the H_{α}, $\alpha \in R$; then $\mathfrak{h}$ is the complexification of $\mathfrak{h}_R$ and V is the dual $\mathfrak{h}_R^*$ of $\mathfrak{h}_R$.

In this way we have constructed from $\mathfrak{g}$ a finite set R of nonzero vectors in the Euclidean space V. This set R is called the *root-system* of $\mathfrak{g}$: up to isomorphism, it is independent of the choice of $\mathfrak{h}$, and therefore depends only on $\mathfrak{g}$. It may be thought of as in some sense the 'skeleton' of $\mathfrak{g}$, and it determines $\mathfrak{g}$ up to isomorphism. More precisely, there is the following *isomorphism theorem*: if $\mathfrak{g}'$ is another complex semisimple Lie algebra; $\mathfrak{h}'$ a Cartan subalgebra of $\mathfrak{g}'$; R' the root system of $\mathfrak{g}'$ relative to $\mathfrak{h}'$; and if

$\phi: \mathfrak{h} \to \mathfrak{h}'$ is an isomorphism which induces a bijection of R' onto R, then ϕ can be extended to a Lie algebra isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(n,C)$ and let $\mathfrak{h}$ be the Cartan subalgebra consisting of the diagonal matrices with trace 0. Let E_{ij} $(1 \le i,j \le n)$ be the matrix with 1 in the (i,j) place and 0 elsewhere; also let ε_i $(1 \le i \le n)$ be the linear form which maps each diagonal matrix to its ith diagonal element. For each $H \in \mathfrak{h}$ we have

$$[H.E_{ij}] = (\varepsilon_i - \varepsilon_j)(H).E_{ij}$$

so that $\varepsilon_i - \varepsilon_j$ is a root of $(\mathfrak{g},\mathfrak{h})$ whenever $i \neq j$; and since

$$\mathfrak{g} = \mathfrak{h} + \sum_{i \neq j} CE_{ij}$$

it follows that these are all the roots. The real space V spanned by the roots has dimension $n-1$. For example, the roots $\varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n-1)$ form a basis of V.

4. Geometry of the root system

For each root $\alpha \in R$ let $w_\alpha: V \to V$ be the reflection in the hyperplane V_α orthogonal to α. Elementary geometry shows that

$$w_\alpha(x) = x - \langle x,\alpha^\vee\rangle\alpha$$

for $x \in V$, where $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ is the *coroot* corresponding to α.

The root system R has the following properties:

(1) $w_\alpha(R) = R$ for each $\alpha \in R$;

(2) $\langle\alpha^\vee,\beta\rangle \in Z$ for each pair $\alpha,\beta \in R$;

(3) if $\alpha,\beta \in R$ are proportional, then $\beta = \pm\alpha$.

We have no space here for the proofs of these various assertions, which may be found in any text on Lie algebras. Let us however briefly indicate the reason for the integrality property (2). For each pair of roots $\pm\alpha$ one can choose root-vectors $X_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $[X_\alpha,X_{-\alpha}] = H_{\alpha^\vee}$, the image of the coroot $\alpha^\vee$ under the isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ induced by the Killing form. The vector space $\mathfrak{s}_\alpha$ spanned by $X_\alpha, X_{-\alpha}$ and $H_{\alpha^\vee}$ is a Lie subalgebra of $\mathfrak{g}$, and the mapping which takes these three vectors respectively to the matrices $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is an isomorphism of $\mathfrak{s}_\alpha$ onto $\mathfrak{sl}(2,C)$. Now a study of the representations of the Lie algebra $\mathfrak{sl}(2,C)$ shows that in any representation ρ the eigenvalues of $\rho\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are *integers*. Since $\langle\alpha^\vee,\beta\rangle = \beta(H_{\alpha^\vee})$ is an eigenvalue of $\mathrm{ad}_{\mathfrak{g}}(H_{\alpha^\vee})$, it follows that $\langle\alpha^\vee,\beta\rangle$ is an integer.

In fact the study of the representations of $\mathfrak{sl}(2,C)$ obtained by restricting $\mathrm{ad}_{\mathfrak{g}}$ to the three-dimensional subalgebras $\mathfrak{s}_\alpha$ is the key to the proofs of the results summarized above.

We can now forget, for the time being, about the Lie algebra $\mathfrak{g}$ and concentrate on the root system R. Abstractly, R can be any finite spanning set of nonzero vectors in a Euclidean space V which satisfies (1),(2),(3) above. The group W generated by the reflections w_α is called the *Weyl group* (of R, or of $\mathfrak{g}$); it acts faithfully as a group of permutations of R, hence is a finite group. Next, the hyperplanes V_α cut up V into congruent open simplicial cones called *chambers*, and a fundamental property of R is that the Weyl group permutes the chambers freely and transitively: that is to say, if we choose a chamber C, then every other chamber is expressible as wC for a unique element $w \in W$. The chamber C is bounded by ℓ $(= \dim V)$ hyperplanes $V_{\alpha_i} = V_{-\alpha_i}$ $(1 \le i \le \ell)$. One of each pair of roots $\pm\alpha_i$, say α_i, is such that $\langle\alpha_i,x\rangle > 0$ for all $x \in C$; the resulting set of ℓ roots $\alpha_1,\ldots,\alpha_\ell$ is called a *basis* of R, or a set of *simple* roots. It can also be characterized by the fact that every root α is a linear combination of the simple roots with integer coefficients, either all ≥ 0 or all ≤ 0. The set of bases of R, being in one-one correspondence with the set of chambers, is permuted freely and transitively by the Weyl group W.

Let θ_{ij} be the angle between the simple roots α_i, α_j. Then

$$\cos^2\theta_{ij} = \langle\alpha_i,\alpha_j\rangle^2/\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle = \frac{1}{4}\langle\alpha_i^\vee,\alpha_j\rangle\langle\alpha_i,\alpha_j^\vee\rangle$$

$= \frac{1}{4} m_{ij}$ say, where $m_{ij} = 0,1,2$ or 3 (since by (2) m_{ij} must be an integer). Since $\theta_{ij} \geq \frac{1}{2}\pi$, the only possible values for the angle θ_{ij} are therefore $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, $\frac{3}{4}\pi$ and $\frac{5}{6}\pi$. If $m_{ij} = 0$, α_i and α_j are orthogonal. If $m_{ij} > 0$, and $\langle\alpha_i,\alpha_i\rangle \geq \langle\alpha_j,\alpha_j\rangle$, then $\langle\alpha_i,\alpha_i\rangle/\langle\alpha_j,\alpha_j\rangle = m_{ij}$.

The relative positions of the simple roots may be described by the *Dynkin diagram*; this is a graph whose vertices are in one-one correspondence with the simple roots, the vertices corresponding to α_i and α_j being joined by m_{ij} bonds and (if $m_{ij} > 1$) an arrow-head pointing (like the inequality sign) towards the *shorter* of α_i and α_j.

An equivalent method of describing the relative positions of the simple roots is the *Cartan matrix*, which is the $\ell\times\ell$ matrix of integers whose (i,j) element is $a_{ij} = \langle\alpha_i^\vee,\alpha_j\rangle$. It has 2's down the diagonal, and its off-diagonal elements are ≤ 0.

The Cartan matrix and the Dynkin diagram each determine the other, and either determines R (and hence $\mathfrak{g}$) up to isomorphism.

A root system R is said to be *irreducible* if there exists no partition of R into two non-empty subsets R_1, R_2 with each root in R_1 orthogonal to each root in R_2; this is the case if and only if the Lie algebra $\mathfrak{g}$ is *simple*. Since two simple roots are orthogonal if and only if the corresponding vertices of the Dynkin diagram are not directly linked, it is not hard

to see that R is irreducible if and only if its Dynkin diagram is *connected*.

Example. If $\mathfrak{g} = \mathfrak{sl}(n,\mathbf{C})$ we may take the simple roots to be $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq n-1)$, in the notation of the Example in §3. The reflection w_α corresponding to the root $\alpha = \varepsilon_i - \varepsilon_j$ interchanges ε_i and ε_j and leaves the remaining ε_k fixed, from which it follows that W is isomorphic to the symmetric group S_n, acting by permuting the ε_i. The Dynkin diagram is a chain

and the Cartan matrix (a_{ij}) has $a_{ij} = 2$ if $i = j$, $a_{ij} = -1$ if $|i-j| = 1$, $a_{ij} = 0$ otherwise.

5. Classification

The classification of the connected Dynkin diagrams is a purely combinatorial undertaking, and leads to the well-known list consisting of the four infinite series A_ℓ $(\ell \geq 1)$, B_ℓ $(\ell \geq 2)$, C_ℓ $(\ell \geq 3)$, D_ℓ $(\ell \geq 4)$ and the five 'exceptional' diagrams E_6, E_7, E_8, F_4, G_2, which will be found in any text on the subject.

Finally, the isomorphism theorem of §3 is complemented by an *existence theorem*, which states that every Dynkin diagram arises from some complex semisimple Lie algebra $\mathfrak{g}$. One constructs $\mathfrak{g}$ by writing down generators and relations, the relations involving only the Cartan integers $a_{ij} = \langle\alpha_i^\vee,\alpha_j\rangle$. From all this it follows that the isomorphism classes of complex simple Lie algebras can be labelled by the same symbols $A_\ell,\ldots,G_2$ used above.

Examples. The simple Lie algebra A_ℓ is $\mathfrak{sl}(\ell+1,\mathbf{C})$. The other 'classical' Lie algebras B_ℓ, C_ℓ, D_ℓ may be briefly described as follows. Let E be a complex vector space of finite dimension n, let f be a nondegenerate symmetric or skew-symmetric bilinear form on E, and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(E)$ consisting of all $X \in \mathfrak{gl}(E)$ such that $f(Xu,v) + f(u,Xv) = 0$ for all $u,v \in E$. Then $\mathfrak{g} = B_\ell$ if $n = 2\ell+1$ and f is symmetric; $\mathfrak{g} = C_\ell$ if $n = 2\ell$ and f is skew-symmetric; and $\mathfrak{g} = D_\ell$ if $n = 2\ell$ and f is symmetric.

In more concrete terms, B_ℓ is $\mathfrak{so}(2\ell+1,C)$ and D_ℓ is $\mathfrak{s}(2\ell,C)$, where $\mathfrak{so}(n,C) \subset \mathfrak{sl}(n,C)$ consists of the skew-symmetric matrices $(X+X^t=0)$; and C_ℓ is $\mathfrak{sp}(2n,C) \subset \mathfrak{sl}(2n,C)$, consisting of the matrices X satisfying $XJ + JX^t = 0$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

6. Real forms

We shall now take up the structure theory of *real* semisimple Lie algebras. Here the situation is more complicated: it can happen (in fact, as we shall see, it always does) that non-isomorphic real Lie algebras have the same (or isomorphic) complexifications. For example the Lie algebras $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,R)$ both have $\mathfrak{sl}(n,C)$ as their complexification.

If $\mathfrak{g}$ is a complex Lie algebra, a real Lie subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$ is a *real form* of $\mathfrak{g}$ if $\mathfrak{g}$ is the complexification of $\mathfrak{g}_o$, i.e. if $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$ (direct sum). Such a real form $\mathfrak{g}_o$ determines a mapping $c\colon \mathfrak{g} \to \mathfrak{g}$, namely $Y+iZ \to Y-iZ$ $(Y,Z \in \mathfrak{g}_o)$. This mapping c has the following properties:

(1) c is *semilinear*, i.e. $c(\lambda X+\mu Y) = \bar\lambda c(X) + \bar\mu c(Y)$ for $X,Y \in \mathfrak{g}$ and $\lambda,\mu \in C$;

(2) c is an *involution*, i.e. $c^2 = 1_{\mathfrak{g}}$;

(3) $c[X,Y] = [cX,cY]$ for $X,Y \in \mathfrak{g}$.

A bijection $c\colon \mathfrak{g} \to \mathfrak{g}$ with these properties is called a *conjugation* of $\mathfrak{g}$. Conversely, any conjugation c of $\mathfrak{g}$ determines uniquely a real subalgebra $\mathfrak{g}_o = \{X \in \mathfrak{g} : cX = X\}$ such that $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$, that is to say a *real form* of $\mathfrak{g}$. Hence we have a canonical one-one correspondence between conjugations

of $\mathfrak{g}$ and real forms of $\mathfrak{g}$.

Now if $\mathfrak{g}_o$ is the Lie algebra of a *compact* Lie group G, the Killing form of $\mathfrak{g}_o$ is negative semi-definite (and negative definite if the centre of $\mathfrak{g}_o$ is zero). For G acts on $\mathfrak{g}$ via the adjoint representation Ad_G; since G is compact, there exists an Ad-invariant positive definite quadratic form Q on $\mathfrak{g}_o$ (take an arbitrary positive definite form, and average it over G). With respect to a Q-orthonormal basis of $\mathfrak{g}_o$, the linear transformation $\mathrm{Ad}_G(x)$ for each $x \in G$ is represented by an *orthogonal* matrix, i.e. we have $\mathrm{Ad}_G\colon G \to O(n)$, where $n = \dim \mathfrak{g}_o$. Hence $\mathrm{ad}_{\mathfrak{g}_o}\colon \mathfrak{g}_o \to \mathfrak{so}(n)$ and therefore each $\mathrm{ad}\,X$ is represented by a skew-symmetric matrix (Chapter I, §5, Examples). Consequently

$$\begin{aligned} B(X,X) &= \mathrm{trace}(\mathrm{ad}\,X)^2 = \sum_{i,j} (\mathrm{ad}\,X)_{ij}(\mathrm{ad}\,X)_{ji} \\ &= -\sum_{i,j} (\mathrm{ad}\,X)_{ij}^2 \le 0, \end{aligned}$$

and $B(X,X) = 0$ if and only if $\mathrm{ad}\,X = 0$, i.e. if and only if X is in the centre of $\mathfrak{g}_o$. For this reason a semisimple real Lie algebra is said to be *compact* if its Killing form is negative definite.

Every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form, which is unique up to isomorphism. It may be constructed as follows: with the notation of §3, vectors $X_\alpha \in \mathfrak{g}^\alpha$ can be chosen for each root α such that for each pair of roots α,β we have

$$[X_\alpha,X_\beta] = \begin{cases} N_{\alpha,\beta}X_{\alpha+\beta} & \text{if } \alpha+\beta \in R, \\ H_\alpha & \text{if } \alpha+\beta = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the constants $N_{\alpha,\beta}$ are *real* and satisfy $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$. From these relations it follows that $B(X_\alpha,X_\beta) = 1$ or 0 according as $\alpha+\beta = 0$ or $\neq 0$. Then the elements iH_α , $X_\alpha - X_{-\alpha}$, $i(X_\alpha + X_{-\alpha})$ span a compact real form of $\mathfrak{g}$.

Another real form of $\mathfrak{g}$ is easily written down, namely the real Lie algebra spanned by the H_α and the X_α . This form is called the *split* (or normal, or anticompact) real form of $\mathfrak{g}$ it is not compact. In a sense to be explained later, these two (the compact and split forms) are at opposite extremes, and in general there will be other real forms as well.

7. Examples: real forms of the classical complex Lie algebras

If $\mathfrak{g} = \mathfrak{sl}(n,C)$, a compact real form of $\mathfrak{g}$ is $\mathfrak{su}(n)$ and a split form is $\mathfrak{sl}(n,R)$. The corresponding conjugations of $\mathfrak{g}$ are respectively $X \to -\bar{X}^t$ and $X \to \bar{X}$.

If n is even, say $n = 2m$, another real form of $\mathfrak{sl}(2m,C)$ is $\mathfrak{sl}(m,H) = \{X \in \mathfrak{gl}(n,H): \mathrm{Re}(\mathrm{trace}\ X) = 0\}$. Any quaternionic matrix may be written as $Y + Zj$, where Y and Z are complex matrices, and we can embed $\mathfrak{sl}(m,H)$ in $\mathfrak{sl}(2m,C)$ by means of the map

$$\eta: Y+Zj \to \begin{pmatrix} Y & Z \\ -\bar{Z} & \bar{Y} \end{pmatrix} ;$$

the image of $\mathfrak{sl}(m,H)$ under η is denoted by $\mathfrak{su}^*(2m)$, and is a real form of $\mathfrak{sl}(2m,C)$. The corresponding conjugation is $X \to J\bar{X}J^{-1}$, where $J = \eta(j)$.

Apart from $\mathfrak{sl}(n,R)$ and $\mathfrak{sl}(m,H)$, the real forms of the classical complex simple Lie algebras $A_\ell, B_\ell, C_\ell, D_\ell$ may all be described uniformly as follows. Let K be any one of R,C,H, and let E be a K-vector space of dimension n (a left vector space if $K = H$). Let $f: E\times E \to K$ be a nondegenerate ε-Hermitian form, where $\varepsilon = \pm 1$ (so that f is K-linear in the first variable, and $f(u,v) = \varepsilon\overline{f(v,u)}$), and let $\mathfrak{g}(E,f) \subset \mathfrak{gl}(E)$ be the subalgebra consisting of all $X \in \mathfrak{gl}(E)$ such that trace $X = 0$ and

$$f(Xu,v) + f(u,Xv) = 0$$

for all $u,v \in E$. Let q denote the Witt index of f (namely the dimension of a maximal totally isotropic subspace of E) and let $p = n-q$, so that $p \geq q \geq 0$ and $p+q = n$. The integers p,q determine f up to isomorphism. The Lie algebras $\mathfrak{g}(E,f)$, for all legitimate choices of K, ε, p and q , together with $\mathfrak{sl}(n,R)$ and $\mathfrak{sl}(n,H)$, exhaust the real forms of the classical complex simple Lie algebras.

(a) Suppose first that $K = R$ and $\varepsilon = +1$. Then f is symmetric, and the algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{so}(p,q)$. It consists of the matrices $X \in \mathfrak{sl}(n,R)$ such that

$$I_{p,q}X + X^t I_{p,q} = 0$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, and is a real form of $\mathfrak{so}(n,C)$. [As it stands, $\mathfrak{so}(p,q)$ is not a subalgebra of $\mathfrak{so}(n,C)$, but the isomorphic algebra $J_{p,q}\mathfrak{so}(p,q)J_{p,q}^{-1}$ is, where $J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & iI_q \end{pmatrix}$; the corresponding conjugation of $\mathfrak{so}(n,C)$ is $X \to I_{p,q}\bar{X}I_{p,q}$.] If $q = 0$, then $\mathfrak{so}(p,q) = \mathfrak{so}(n,0) = \mathfrak{so}(n)$ is the compact real form of $\mathfrak{so}(n,C)$. At the other extreme, $q = [\frac{1}{2}n]$ gives the split form.

(b) Next suppose that $K = R$, $\varepsilon = -1$. Then f is skew-symmetric and hence (because f is nondegenerate) n is even, say $n = 2m$, and the index q is equal to m . The algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{sp}(2m,R)$: it consists of the matrices $X \in \mathfrak{sl}(2m,R)$ satisfying $XJ + JX^t = 0$. Hence it is the split real form of $\mathfrak{sp}(2m,C)$, the conjugation being $X \to \bar{X}$.

(c) Now let $K = C$. Here we do not need to distinguish between $\varepsilon = +1$ and $\varepsilon = -1$, because if f is antihermitian then if is hermitian. We may therefore assume that f is *hermitian.* The algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{su}(p,q)$: it consists of the matrices $X \in \mathfrak{sl}(n,C)$ such that

$$(*) \qquad I_{p,q}X + \bar{X}^t I_{p,q} = 0$$

and is a real form of $\mathfrak{sl}(n,C)$, the conjugation being $X \to -I_{p,q}\bar{X}^t I_{p,q}$. When $q = 0$, we have $\mathfrak{su}(p,q) = \mathfrak{su}(n,0) = \mathfrak{su}(n)$ the compact real form of $\mathfrak{sl}(n,C)$.

(d) Let $K = H$, $\varepsilon = +1$. Then f is a quaternionic Hermitian form, and the Lie algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{sp}(p,q)$. It consists of the matrices $X \in \mathfrak{gl}(n,H)$ satisfying (*), and under the embedding η of $\mathfrak{gl}(n,H)$ in $\mathfrak{gl}(2n,C)$ it is a real form of $\mathfrak{sp}(2n,C)$. When $q = 0$, we have $\mathfrak{sp}(p,q) = \mathfrak{sp}(n,0) = \mathfrak{sp}(n) = \mathfrak{u}(n,H)$, the Lie algebra of the compact group $Sp(n) = U(n,H)$, which is therefore the compact real form of $\mathfrak{sp}(2n,C)$.

(e) Finally, let $K = H$ and $\varepsilon = -1$. Then f is quaternionic antihermitian, which since f is nondegenerate implies that the index q is $[\frac{1}{2}n]$. The corresponding Lie algebra $\mathfrak{g}(E,f)$ may be taken to consist of the matrices $X \in \mathfrak{sl}(n,H)$ such that $Xj + j\bar{X}^t = 0$; it is denoted by $\mathfrak{s\alpha u}(n,H)$. Its image in $\mathfrak{gl}(2n,C)$ under η is a subalgebra $\mathfrak{so}^*(2n)$ of $\mathfrak{so}(2n,C)$, consisting of the $X \in \mathfrak{so}(2n,C)$ such that $XJ + J\bar{X}^t = 0$, and is a real form of $\mathfrak{so}(2n,C)$, the conjugation being $X \to J\bar{X}^t J$.

To summarize, the real forms of A_ℓ $(\ell \geq 1)$ are

$$A_\ell^R = \mathfrak{sl}(\ell+1,R)$$

$$A_\ell^{C,q} = \mathfrak{su}(p,q)$$

(where $p \geq q \geq 0$ and $p+q = \ell+1$, so that $0 \leq q \leq [\frac{1}{2}(\ell+1)]$)

$$A_\ell^H = \mathfrak{sl}(m+1,H) \quad (\cong \mathfrak{su}^*(2m))$$

(if $\ell = 2m-1$ is odd).

The split form is A_ℓ^R and the compact form is $A_\ell^{C,0} = \mathfrak{su}(\ell+1)$.

The real forms of B_ℓ $(\ell \geq 2)$ are

$$B_\ell^{R,q} = \mathfrak{so}(p,q)$$

(where $p \geq q \geq 0$ and $p+q = 2\ell+1$, so that $0 \leq q \leq \ell$) .

The split form is $\mathfrak{so}(\ell+1,\ell)$ $(q = \ell)$ and the compact form is $\mathfrak{so}(2\ell+1)$ $(q = 0)$.

The real forms of C_ℓ $(\ell \geq 3)$ are

$$C_\ell^R = \mathfrak{sp}(2\ell,R)$$

$$C_\ell^{H,q} = \mathfrak{sp}(p,q)$$

(where $p \geq q \geq 0$ and $p+q = \ell$, so that $0 \leq q \leq [\frac{1}{2}\ell]$).

The split form is C_ℓ^R and the compact form is $C_\ell^{H,0} = \mathfrak{sp}(\ell) = \mathfrak{u}(\ell,H)$.

The real forms of D_ℓ $(\ell \geq 4)$ are

$$D_\ell^{R,q} = \mathfrak{so}(p,q)$$

(where $p \geq q \geq 0$ and $p+q = 2\ell$, so that $0 \leq q \leq \ell$)

$$D_\ell^H = \mathfrak{s\alpha u}(\ell,H) \quad (\cong \mathfrak{so}^*(2\ell))$$

The split form is $\mathfrak{so}(\ell,\ell)$ and the compact form is $\mathfrak{so}(2\ell)$. (When $\ell = 4$, we have $D_4^H = D_4^{R,2}$.)

8. The Cartan decomposition

Let us return to the general theory. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{u}$ a compact real form of $\mathfrak{g}$, and

c_u the conjugation (§6) of $\mathfrak{g}$ defined by $\mathfrak{u}$. If c is any conjugation of $\mathfrak{g}$, there exists an automorphism ϕ of $\mathfrak{g}$ such that c_u commutes with $\phi c \phi^{-1}$; hence to find all real forms of $\mathfrak{g}$, up to isomorphism, it is enough to find all conjugations c of $\mathfrak{g}$ which commute with c_u.

If c commutes with c_u, we have $c(\mathfrak{u}) = \mathfrak{u}$ and $c(i\mathfrak{u}) = i\mathfrak{u}$. Let $\mathfrak{k}$ and $i\mathfrak{p}$ be respectively the $+1$ and -1 eigenspaces of c in $\mathfrak{u}$, so that

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$$

(direct sum), and the $+1$ and -1 eigenspaces of c on $i\mathfrak{u}$ are $\mathfrak{p}$ and $i\mathfrak{k}$, so that

$$i\mathfrak{u} = \mathfrak{p} + i\mathfrak{k}$$

Hence if $\mathfrak{g}_o$ is the real form of $\mathfrak{g}$ determined by c, we have

$$(*) \quad \mathfrak{g}_o = \mathfrak{k} + \mathfrak{p} ;$$

$\mathfrak{k} = \mathfrak{g}_o \cap \mathfrak{u}$ is a subalgebra of $\mathfrak{g}_o$ and $\mathfrak{p} = \mathfrak{g}_o \cap i\mathfrak{u}$ is a vector subspace (*not* a subalgebra) such that

(i) the Killing form $B_{\mathfrak{g}_o}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$;

(ii) the map $cc_u = c_u c = \theta : Y+Z \to Y-Z \quad (Y \in \mathfrak{k},\ Z \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}_o$.

A direct decomposition (*) of $\mathfrak{g}_o$, constructed as above from a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ such that c_u commutes with c, is called a *Cartan decomposition* of the real Lie algebra $\mathfrak{g}_o$, and θ is a *Cartan involution* of $\mathfrak{g}_o$. The Cartan decomposition is determined by the involution θ, since $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ in $\mathfrak{g}_o$. We have $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with

respect to $B_{\mathfrak{g}_o}$.

If θ is *any* involutory automorphism of $\mathfrak{g}_o$, the bilinear form $\langle X,Y\rangle_\theta = -B_{\mathfrak{g}_o}(X,\theta Y)$ is symmetric, and θ is a Cartan involution if and only if $\langle X,X\rangle_\theta$ is positive definite.

The importance of the Cartan decomposition is that it is unique up to conjugacy: if $\mathfrak{g}_o = \mathfrak{k}' + \mathfrak{p}'$ is another Cartan decomposition, there exists $\phi \in \mathrm{Int}(\mathfrak{g}_o)$ such that $\mathfrak{k}' = \phi(\mathfrak{k})$ and $\mathfrak{p}' = \phi(\mathfrak{p})$.

Define the *Cartan signature* (some say *Cartan index*) s of $\mathfrak{g}_o$ to be the signature of the real quadratic form $B_{\mathfrak{g}_o}$, i.e.

$$s = \dim \mathfrak{p} - \dim \mathfrak{k} .$$

Then we have

$$-\dim \mathfrak{g} \le s \le \operatorname{rank} \mathfrak{g}$$

and $s = -\dim \mathfrak{g} \iff \mathfrak{g}_o$ is compact,

$$s = \operatorname{rank} \mathfrak{g} \iff \mathfrak{g}_o \text{ is split.}$$

Examples

1. For the compact real form we have $\mathfrak{p} = 0$, $\mathfrak{k} = \mathfrak{g}_o$, and θ is the identity map. For the split real form of $\mathfrak{g}$, spanned by the H_α and the X_α (§6), $\mathfrak{k}$ is spanned by the $X_\alpha - X_{-\alpha}$ and $\mathfrak{p}$ is spanned by the H_α and the $X_\alpha + X_{-\alpha}$.
2. For the real forms of the classical complex Lie algebras listed in §7, in each case $\theta : X \to -\bar{X}^t$ ($= -X^t$ if X is real) is a Cartan involution. Hence $\mathfrak{k}$ consists of the skew Hermitian matrices in $\mathfrak{g}$, and $\mathfrak{p}$ consists of the Hermitian matrices.

9. The Iwasawa decomposition

From now on the emphasis will be on a fixed real semisimple Lie algebra, which we shall denote by $\mathfrak{g}$ (rather than $\mathfrak{g}_o$);

the complexification of $\mathfrak{g}$, which is a complex semisimple Lie algebra, will be denoted by $\mathfrak{g}_c$ (instead of $\mathfrak{g}$ as heretofore). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition of $\mathfrak{g}$, and let θ be the associated Cartan involution, so that θ is the identity on $\mathfrak{k}$, and minus the identity on $\mathfrak{p}$. The bilinear form on $\mathfrak{g}$

$$\langle X,Y\rangle_\theta = -B_{\mathfrak{g}}(X,\theta Y)$$

is symmetric and positive definite (it coincides with $B_{\mathfrak{g}}$ on $\mathfrak{p}$ and with $-B_{\mathfrak{g}}$ on $\mathfrak{k}$), hence endows $\mathfrak{g}$ with the structure of a finite-dimensional real Hilbert space. For any $X \in \mathfrak{g}$, the adjoint of ad X (with respect to this scalar product) is $-\mathrm{ad}\,\theta(X)$. Hence, with respect to an orthonormal basis of $\mathfrak{g}$, ad X is represented by a symmetric matrix if $X \in \mathfrak{p}$, and by a skew-symmetric matrix if $X \in \mathfrak{k}$. It follows that the elements of $\mathfrak{k}$ and $\mathfrak{p}$ are semisimple.

Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subalgebra of the vector space $\mathfrak{p}$, and let $\mathfrak{a}_\mathfrak{p}^*$ be the vector space dual to $\mathfrak{a}_\mathfrak{p}$. For each $\lambda \in \mathfrak{a}_\mathfrak{p}^*$ let

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}_\mathfrak{p}\} .$$

Since $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a}_\mathfrak{p})$ is a commuting family of self-adjoint linear transformations of $\mathfrak{g}$, it follows that $\mathfrak{g}$ is the orthogonal direct sum of the subspaces $\mathfrak{g}^\lambda$. If $\lambda \neq 0$ and $\mathfrak{g}^\lambda \neq 0$, λ is said to be a *root* of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$, and $\mathfrak{g}^\lambda$ is the *root-space* corresponding to λ. Let $S = S(\mathfrak{g},\mathfrak{a}_\mathfrak{p})$ denote the set of roots. Then we have a *root space decomposition*

$$\mathfrak{g} = \mathfrak{l} + \sum_{\lambda\in S} \mathfrak{g}^\lambda$$

(orthogonal direct sum), in which $\mathfrak{l}$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{g}$. The set $S \subset \mathfrak{a}_\mathfrak{p}^*$ is called the *relative root system* of $\mathfrak{g}$ (with respect to $\mathfrak{a}_\mathfrak{p}$); up to isomorphism, it depends only on $\mathfrak{g}$. However, there are divergences from the complex case considered in §3. First, S is a root system in $\mathfrak{a}_\mathfrak{p}^*$, in the sense of §4, but need not be reduced (i.e. need not satisfy condition (3) of §4: this means that it may happen that $\lambda \in S$ and $2\lambda \in S$). Secondly, the dimension m_λ of $\mathfrak{g}^\lambda$ (the *multiplicity* of $\lambda \in S$) may be bigger than 1. Thirdly, $\mathfrak{l}$ is usually bigger than $\mathfrak{a}_\mathfrak{p}$: in fact we have

$$\mathfrak{l} = \mathfrak{m} + \mathfrak{a}_\mathfrak{p}$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. Finally, the root system S does not of itself determine $\mathfrak{g}$ up to isomorphism; for this purpose we require a more elaborate combinatorial object, which we shall describe in the next section.

Choose a basis of S, and let S^+ be the set of positive roots relative to this basis; and write

$$\mathfrak{n} = \sum_{\lambda\in S^+} \mathfrak{g}^\lambda , \quad \bar{\mathfrak{n}} = \sum_{\lambda\in S^+} \mathfrak{g}^{-\lambda}.$$

We have $\theta(\mathfrak{g}^\lambda) = \mathfrak{g}^{-\lambda}$ for all $\lambda \in S$ (because θ acts as -1 on $\mathfrak{a}_\mathfrak{p}$) and therefore $\theta\mathfrak{n} = \bar{\mathfrak{n}}$. $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are nilpotent subalgebras of $\mathfrak{g}$, and we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$$

(direct sum); this is the *Iwasawa decomposition* of $\mathfrak{g}$. If $\mathfrak{s} = \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$, then $\mathfrak{s}$ is a solvable subalgebra of $\mathfrak{g}$.

In Chapter III we shall see that the Iwasawa decomposition has a global counterpart, for any connected semisimple Lie group.

10. The relative root system

We retain the notation of §9. To get more insight into the relative root system S, we shall compare it with the (absolute) root system R of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$, relative to a suitably chosen Cartan subalgebra. For this purpose let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_\mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$; we have $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$, and if we put $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$ then

$$\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$$

(direct sum), so that $\mathfrak{a}$ is stable under θ. The dimension of $\mathfrak{a}_\mathfrak{p}$ is called the *relative rank* (or *split rank*) of $\mathfrak{g}$.

The complexification $\mathfrak{h} = \mathfrak{a}_c$ of $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_c$. Hence (§3) we have a root-space decomposition of $\mathfrak{g}_c$ relative to $\mathfrak{h}$:

$$\mathfrak{g}_c = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_c^\alpha$$

where $R = R(\mathfrak{g}_c, \mathfrak{h}) \subset \mathfrak{h}_R^*$ is the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{h}$. Here $\mathfrak{h}_R$ is the real vector space spanned by the H_α, $\alpha \in R$, as in §3, and in fact

$$\mathfrak{h}_R = \mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k} .$$

Moreover, $i\mathfrak{h}_R = \mathfrak{a}_\mathfrak{k} + i\,\mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of the compact form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_c$.

Let $\rho : \mathfrak{h}_R^* \to \mathfrak{a}_\mathfrak{p}^*$ denote restriction to $\mathfrak{a}_\mathfrak{p}$. The kernel of ρ may be identified with $(i\mathfrak{a}_\mathfrak{k})^*$. For each $\alpha \in R$, the restriction $\rho(\alpha)$ of α to $\mathfrak{a}_\mathfrak{p}$ is either 0 or is an element of S. Let R_o be the set of roots $\alpha \in R$ which vanish on $\mathfrak{a}_\mathfrak{p}$, or equivalently such that $H_\alpha \in i\,\mathfrak{a}_\mathfrak{k}$. Then R_o is a root system in $(i\mathfrak{a}_\mathfrak{k})^*$ (except that it may span a proper subspace of $(i\mathfrak{a}_\mathfrak{k})^*$), and is the root system of the complex reductive Lie algebra $\mathfrak{m}_c$ (where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$) relative to its Cartan subalgebra $(\mathfrak{a}_\mathfrak{k})_c$: we have another root-space decomposition

$$\mathfrak{m}_c = (\mathfrak{a}_\mathfrak{k})_c + \sum_{\alpha \in R_o} \mathfrak{g}_c^\alpha .$$

The projection ρ maps $R-R_o$ onto S, and for each $\lambda \in S$ the multiplicity m_λ is equal to the number of roots $\alpha \in R-R_o$ such that $\rho(\alpha) = \lambda$.

Let c be the conjugation of $\mathfrak{g}_c$ defined by $\mathfrak{g}$, so that $c(X+iY) = X-iY$ for $X,Y \in \mathfrak{g}$. c acts on the root spaces as follows: for each root $\alpha \in R$ define α^σ by

$$\alpha^\sigma(H) = \overline{\alpha(c(H))} \qquad (H \in \mathfrak{h}) .$$

Then $c(\mathfrak{g}_c^\alpha) = \mathfrak{g}_c^{\alpha^\sigma}$. The mapping $\alpha \to \alpha^\sigma$ extends by linearity to an involutory isometry of the Euclidean space $\mathfrak{h}_R^*$, under which $R-R_o$ is stable, and R_o is the set of roots $\alpha \in R$ such that $\alpha^\sigma+\alpha = 0$. We have $\alpha^\sigma-\alpha \notin R$ for all $\alpha \in R$.

Abstractly, therefore, we are led to consider pairs (R,σ), where R is a reduced root system in a Euclidean space V, and σ is an involutory isometry of V such that $\sigma(R) = R$. The pair (R,σ) is said to be *normal* if $\alpha \in R \Rightarrow \alpha^\sigma-\alpha \notin R$.

Let $\rho = \frac{1}{2}(1+\sigma)$, so that ρ is the orthogonal projection of V on $V_1 = V^\sigma$ with kernel $V_o = V^{-\sigma}$. Let $R_o = R \cap V_o$ and $S = \rho(R-R_o)$. Then R_o is a reduced root system in V_o (but may span a proper subspace of V_o) and S is a (not necessarily reduced) root system in V_1. We can choose a basis Γ of R such that $\Gamma_o = \Gamma \cap R_o$ is a basis of R_o, and such that $R^+-R_o^+$ is σ-stable, where R^+ and R_o^+ are the sets of positive roots determined by Γ and Γ_o respectively. The involution σ determines an involutory permutation of $\Gamma-\Gamma_o$ as follows: if $\alpha \in \Gamma-\Gamma_o$, there exists a unique $\beta \in \Gamma-\Gamma_o$ such that $\alpha^\sigma \equiv \beta \pmod{\mathbb{Z}\Gamma_o}$, and the mapping $\alpha \to \beta$ is a permutation of order 2. We have $\rho(\alpha) = \rho(\beta) = \frac{1}{2}(\alpha+\beta)$, and $\Delta = \rho(\Gamma-\Gamma_o)$ is a basis of the root system S. Finally, if $W(R)$, $W(R_o)$ and $W(S)$ are the Weyl groups of the root systems R, R_o, S, and if $W(R)^\sigma$ is the centralizer of σ in $W(R)$, then restriction to V_1 defines a homomorphism of $W(R)^\sigma$ onto $W(S)$, the kernel being $W(R_o)$.

Each real semisimple Lie algebra $\mathfrak{g}$ therefore determines a normal pair (R,σ), which determines $\mathfrak{g}$ up to isomorphism. As described in §4, the reduced root system R may be represented by its Dynkin diagram, the vertices of which represent the elements of the basis Γ of R. The action of σ may be indicated as follows: the vertices of the diagram which represent the elements of Γ_o are coloured black, the remainder white, and two white vertices representing elements $\alpha, \beta \in \Gamma-\Gamma_o$ as above, such that $\rho(\alpha) = \rho(\beta)$, are joined by an arrow $\curvearrowleft\!\curvearrowright$. The resulting diagram is called the *Satake diagram* of $\mathfrak{g}$, and determines $\mathfrak{g}$ up to isomorphism.

Examples

1. If $\mathfrak{g}$ is compact we have $\mathfrak{g} = \mathfrak{k}$, $\mathfrak{p} = 0$, $\mathfrak{a}_\mathfrak{p} = 0$, so that $R_o = R$ and $S = \emptyset$. In this case all the vertices of the Satake diagram of $\mathfrak{g}$ are *black*.

2. At the other extreme, if $\mathfrak{g}$ is split, we may take $\mathfrak{p}$ to be the vector subspace spanned by the H_α and the $X_\alpha+X_{-\alpha}$ (§8, Ex.1), and $\mathfrak{a}_\mathfrak{p}$ to be spanned by the H_α; thus $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}$ and $\mathfrak{a}_\mathfrak{k} = 0$, so that $R_o = \emptyset$ and σ is the identity. In this case all the vertices of the Satake diagram of $\mathfrak{g}$ are *white*, and it coincides with the Dynkin diagram of R (or $\mathfrak{g}_c$).

3. Let $\mathfrak{g}_1$ be a complex semisimple Lie algebra, $\mathfrak{g} = \mathfrak{g}_1^R$ its realification (§1). Multiplication by i is an endomorphism of $\mathfrak{g}$ satisfying $i^2 = -1$ and $[X,iY] = [iX,Y] = i[X,Y]$ for all $X,Y \in \mathfrak{g}$. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{k}+i\mathfrak{k}$ is a Cartan decomposition of $\mathfrak{g}$. We may take $\mathfrak{a}_\mathfrak{p} = i\mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$; then $\mathfrak{a} = \mathfrak{t}+i\mathfrak{t}$

is a Cartan subalgebra of $\mathfrak{g}$, and is the realification of a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}_1$. The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}_\mathfrak{p}$ is then the same as the root space decomposition of $\mathfrak{g}_1$ with respect to $\mathfrak{h}_1$. Hence if we denote by R_1 the set of roots of $(\mathfrak{g}_1,\mathfrak{h}_1)$, we have $R = R_1 \times R_1$, $R_o = \emptyset$ and $S = R_1$. The Satake diagram of $\mathfrak{g}$ therefore consists of two copies of the Dynkin diagram of R_1, corresponding vertices in the two copies being joined by arrows.

4. For a concrete example not covered by Exx.1-3, consider $\mathfrak{g} = \mathfrak{sl}(n,H)$ (§7). Then (§8, Ex.2) we may take $\mathfrak{k} = \mathfrak{sp}(n)$ $(= \mathfrak{u}(n,H))$ and $\mathfrak{p}$ to consist of the quaternionic Hermitian matrices with trace 0. We may then take $\mathfrak{a}_\mathfrak{p}$ to consist of the real diagonal matrices with trace 0, and $\mathfrak{a}$ to consist of the complex diagonal matrices X with $\mathrm{Re}(\mathrm{trace}\ X) = 0$. If we embed $\mathfrak{sl}(n,H)$ in $\mathfrak{sl}(2n,C)$ as in §7, then $\mathfrak{a}_c = \mathfrak{h}$ consists of the diagonal matrices in $\mathfrak{sl}(2n,C)$, and we have $R = \{\varepsilon_i-\varepsilon_j : i \neq j,\ 1 \leq i,j \leq 2n\}$ in the notation of §3, Example. Here $R_o = \{\varepsilon_i-\varepsilon_j : |i-j| = n\}$; hence if we put $\varepsilon_i' = \varepsilon_{i\pm n}$, we may take

$$\Gamma = \{\varepsilon_1'-\varepsilon_1, \varepsilon_1-\varepsilon_2', \varepsilon_2'-\varepsilon_2, \dots, \varepsilon_{n-1}-\varepsilon_n', \varepsilon_n'-\varepsilon_n\},$$

$$\Gamma_o = \{\varepsilon_1'-\varepsilon_1, \varepsilon_2'-\varepsilon_2, \dots, \varepsilon_n'-\varepsilon_n\}$$

and the Satake diagram is

there being n black vertices and $n-1$ white ones. The relative root system is non-reduced, of type BC_{n-1}.

5. For another example, take $\mathfrak{g} = \mathfrak{su}(p,q)$, consisting of all complex matrices of the form

$$\begin{pmatrix} A & B \\ \bar{B}^t & C \end{pmatrix}$$

where $A \in \mathfrak{u}(p)$, $B \in \mathfrak{u}(q)$, trace A + trace $C = 0$, and B is any $p\times q$ complex matrix. As in §7, we shall assume $p \geq q$.

Let $H(X,Y,Z)$ denote the matrix

$$\begin{pmatrix} X & 0 & Y \\ 0 & Z & 0 \\ Y & 0 & X \end{pmatrix}$$

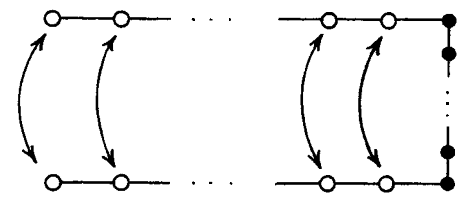
where X,Y,Z are diagonal matrices of sizes $q,p-q,q$ respectively. We may take $\mathfrak{a}_\mathfrak{p}$ to consist of all $H(0,Y,0)$ with Y a real diagonal matrix; $\mathfrak{a}_\mathfrak{k}$ to consist of all $H(iX,0,iZ)$ with X,Z diagonal matrices, and trace $H = 0$. Then $\mathfrak{h} = \mathfrak{a}_c$ consists of all $H(X,Y,Z)$ with X,Y,Z complex diagonal matrices (and trace $H = 0$). Conjugation by the matrix $\begin{pmatrix} \alpha & 0 & \alpha \\ 0 & I & 0 \\ -\alpha & 0 & \alpha \end{pmatrix}$, where $\alpha = \frac{1}{\sqrt{2}} I_q$, transforms $\mathfrak{h}$ into the standard Cartan subalgebra of $\mathfrak{g}_c = \mathfrak{sl}(p+q,C)$. Hence if ξ_j (resp. η_j, resp. ζ_j) is the linear form on $\mathfrak{h}$ whose value at $H(X,Y,Z)$ is the jth diagonal element of X (resp. Y, resp. Z), then the roots of $(\mathfrak{g}_c,\mathfrak{h})$ are the differences between pairs of

$$\xi_1+\eta_1,\ldots,\xi_q+\eta_q,\zeta_1,\ldots,\zeta_{p-q},\xi_q-\eta_q,\ldots,\xi_1-\eta_1 .$$

We may take the basis Γ to consist of the differences of consecutive pairs of these $p+q$ elements of $\mathfrak{h}^*$. On restriction to $\mathfrak{a}_\mathfrak{p}$, all ξ_j and ζ_k vanish; if $\tilde{\eta}_j$ is the restriction of η_j to $\mathfrak{a}_\mathfrak{p}$, the restrictions to $\mathfrak{a}_\mathfrak{p}$ of the elements of Γ are therefore

$$\tilde{\eta}_1-\tilde{\eta}_2,\ldots,\tilde{\eta}_{q-1}-\tilde{\eta}_q,\tilde{\eta}_q,0,\ldots,0,\tilde{\eta}_q,\tilde{\eta}_{q-1}-\tilde{\eta}_q,\ldots,\tilde{\eta}_1-\tilde{\eta}_2 .$$

It follows that the Satake diagram is



there being q pairs of white vertices and $p-q-1$ black vertices. The root system S is of type BC_q.

11. Cartan subalgebras again

Let $\mathfrak{g}$ be a real semisimple Lie algebra. By contrast with the complex case, as we have already observed, it is no longer true in general that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (unless $\mathfrak{g}$ is compact); instead, they form a finite number of conjugacy classes. They all have the same dimension (namely rank $\mathfrak{g}$), because their complexifications are Cartan subalgebras of $\mathfrak{g}_c$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(2,R)$ and let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be the subspace of $\mathfrak{g}$ spanned by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). Then $\mathfrak{a}$ and $\mathfrak{b}$ are Cartan subalgebras of $\mathfrak{g}$. They cannot be conjugate in $\mathfrak{g}$, because the subgroup $\exp(\mathfrak{a})$ of $SL(2,R)$ consists of all matrices $\begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$, hence is isomorphic to the additive group R, and in particular is not compact; whereas $\exp(\mathfrak{b})$ consists of all matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, hence is $SO(2)$ and therefore compact.

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and let θ be the associated Cartan involution. As in §10, let $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$ is a maximal abelian subalgebra of the vector space $\mathfrak{p}$.

If now $\mathfrak{b}$ is any Cartan subalgebra of $\mathfrak{g}$, there exists a conjugate of $\mathfrak{b}$ which is θ-stable, i.e. such that

$$\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} + \mathfrak{b}_{\mathfrak{p}}$$

where $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{b} \cap \mathfrak{k}$ and $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{b} \cap \mathfrak{p}$. The component $\mathfrak{b}_{\mathfrak{p}}$ is called the *toral part* (or the compact part), and $\mathfrak{b}_{\mathfrak{p}}$ the *vector part* of $\mathfrak{b}$. The vector part $\mathfrak{b}_{\mathfrak{p}}$ is an abelian subalgebra of $\mathfrak{p}$, hence is contained in a maximal abelian subalgebra of $\mathfrak{p}$. By conjugating $\mathfrak{b}$, we may arrange that $\mathfrak{b}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$, and then by conjugating again, leaving $\mathfrak{b}_{\mathfrak{p}}$ fixed, we can also arrange that $\mathfrak{b}_{\mathfrak{k}} \supset \mathfrak{a}_{\mathfrak{k}}$. The Cartan subalgebra $\mathfrak{b}$ is said to be *standard* (relative to θ and $\mathfrak{a}$) if these conditions are satisfied.

The classification of Cartan subalgebras up to conjugacy in $\mathfrak{g}$ can now be reduced to a combinatorial problem, as follows. A subset E of the root system R (§10) is said to be *strongly orthogonal* if $\alpha\pm\beta \notin R$ for all pairs $\alpha,\beta \in E$. Now let $\mathfrak{b}$ be a standard Cartan subalgebra of $\mathfrak{g}$. Then there exists a strongly orthogonal set E in R such that the vectors H_α, $\alpha \in E$, form a basis of the orthogonal complement of $\mathfrak{b}_{\mathfrak{p}}$ in $\mathfrak{a}_{\mathfrak{p}}$; moreover E is determined by $\mathfrak{b}$ up to conjugacy by the Weyl group W of R. In this way one establishes a one-one correspondence between the conjugacy classes of Cartan subalgebras in $\mathfrak{g}$ and the W-orbits of strongly orthogonal subsets of the set of roots $\alpha \in R$ such that $H_\alpha \in \mathfrak{a}_{\mathfrak{p}}$.

The two extreme cases are:

(i) the vector part $\mathfrak{b}_{\mathfrak{p}}$ of $\mathfrak{b}$ is as large as possible: if $\mathfrak{b}$ is standard, this means that $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$, hence $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}}$ and consequently $\mathfrak{b} = \mathfrak{a}$. These are the *minimally compact* Cartan subalgebras of $\mathfrak{g}$; they are all conjugate, and they correspond to $E = \emptyset$ above.

(ii) the toral part $\mathfrak{b}_{\mathfrak{k}}$ of $\mathfrak{b}$ is as large as possible: this means that $\mathfrak{b}_{\mathfrak{k}}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{k}$. These are the *maximally compact* (or fundamental) Cartan subalgebras of $\mathfrak{g}$; again they form a single conjugacy class. If $\mathfrak{g}$ itself is compact, they are the only ones.

Examples

1. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{k} = \mathfrak{su}(n)$, $\mathfrak{p}$ the space of real symmetric $n\times n$ matrices with zero trace; $\mathfrak{a} = \mathfrak{a}_{\mathfrak{p}}$ is the space of diagonal matrices with trace 0, and $\mathfrak{a}_{\mathfrak{k}} = 0$. Any strongly orthogonal set E is, up to conjugacy by $W = S_n$, of the form

$$E = \{\varepsilon_1-\varepsilon_2, \varepsilon_3-\varepsilon_4, \ldots, \varepsilon_{2k-1}-\varepsilon_{2k}\}$$

where $k \le \frac{1}{2}n$. Hence the number of conjugacy classes of Cartan subalgebras in $\mathfrak{sl}(n,\mathbb{R})$ is $1 + [\frac{1}{2}n]$. For $0 \le k \le [\frac{1}{2}n]$, let $\mathfrak{b}^{(k)}$ denote the set of matrices in $\mathfrak{sl}(n,\mathbb{R})$ which are of the form

$$\operatorname{diag}(X_1, X_2, \ldots, X_k, Y)$$

where $X_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ and Y is a diagonal matrix of size $n-2k$. The $\mathfrak{b}^{(k)}$ are representatives of the classes of Cartan subalgebras.

2. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{H})$. Here all the Cartan subalgebras are conjugate to the algebra $\mathfrak{a}$ described in §10, Ex.3, since $\mathfrak{a}$ is both maximally and minimally compact.

3. $\mathfrak{g} = \mathfrak{su}(p,q)$ $(p \ge q)$. Here there are $q+1$ conjugacy classes of Cartan subalgebras, representatives of which may be described as follows. With the notation of §10, Ex.4, for each $j = 0,1,\ldots,q$ let $\mathfrak{b}^{(j)}$ consist of all matrices in $\mathfrak{g}$ of the form

$$\begin{pmatrix} iX & 0 & Y \\ 0 & iZ & 0 \\ Y & 0 & iX \end{pmatrix},$$

where X,Y,Z are real diagonal matrices of sizes $q-j$, $q-j$, $p-q+2j$ respectively. Then $\mathfrak{b}^{(0)},\ldots,\mathfrak{b}^{(q)}$ are $q+1$ nonconjugate Cartan subalgebras of $\mathfrak{g} = \mathfrak{su}(p,q)$; $\mathfrak{b}^{(o)}$ is minimally compact and $\mathfrak{b}^{(q)}$ is maximally compact.

III. Semisimple Lie groups

1. Semisimple and reductive Lie groups

Let G be a Lie group. The largest connected solvable normal subgroup of G is called the *radical* R of G. It is a closed subgroup of G, and its Lie algebra is the radical (II, §1) of the Lie algebra $\mathfrak{g}$ of G.

A *connected* Lie group G is said to be *semisimple* if $R = \{e\}$, or equivalently if its Lie algebra $\mathfrak{g}$ is semisimple. Every semisimple Lie group G is equal to its derived group $[G,G]$, and the centre of G is discrete. If $\mathfrak{g}$ is *simple*, G is said to be *almost simple*. A connected and simply-connected Lie group G is semisimple if and only if it is a direct product of almost simple groups.

Finally, a connected Lie group is said to be *reductive* if its Lie algebra is reductive. Every compact connected Lie group is reductive.

Examples. $SL(n,\mathbf{R})$, $SU(n)$ are semisimple (indeed almost simple); $GL^+(n,\mathbf{R})$, $U(n)$ are reductive but not semisimple.

2. Cartan and Iwasawa decompositions

Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition (II, §8) of $\mathfrak{g}$. The immersed subgroup K of G which corresponds to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ (II, §5) is then *closed* in G, and $\exp_G(\mathfrak{p}) = P$ say is a

closed submanifold of G (*not* a subgroup). We have

$$G = K.P$$

and more precisely the mapping $(x,Y) \to x.\exp Y$ is a diffeomorphism of $K\times \mathfrak{p}$ onto G. This is the *Cartan decomposition* of G : it is the global counterpart of the Cartan decomposition of $\mathfrak{g}$. The mapping $\theta : xy \to xy^{-1}$ $(x \in K,\ y \in P)$ is an involutory automorphism of G, and K is its group of fixed points.

The group K is compact if and only if the centre of G is finite. In general we have $K = K_o\times V$ where K_o is a maximal compact subgroup of G (necessarily connected), and V is a vector group. It follows that G is diffeomorphic to the product of K_o and a vector group, and therefore $\pi_1(G) = \pi_1(K_o)$.

Next let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_{\mathfrak{p}} + \mathfrak{n}$$

be an Iwasawa decomposition (II, §9) of $\mathfrak{g}$, and let $A_{\mathfrak{p}}$ and N be the immersed subgroups of G which correspond to the subalgebras $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{n}$ of $\mathfrak{g}$. Then $A_{\mathfrak{p}}$ and N are closed subgroups of G, and the exponential map $\exp_G$ is an isomorphism of $\mathfrak{a}_{\mathfrak{p}}$ onto $A_{\mathfrak{p}}$, and a diffeomorphism of $\mathfrak{n}$ onto N; $A_{\mathfrak{p}}$ is a vector group and N is a nilpotent Lie group. We have

$$G = KA_{\mathfrak{p}}N$$

and more precisely the mapping $(x,y,z) \to xyz$ is a diffeomorphism of $K\times A_{\mathfrak{p}}\times N$ onto G. Finally, $A_{\mathfrak{p}}N$ is a closed solvable subgroup of G in which N is normal. This is the *Iwasawa decomposition* of G ; it is the global counterpart of the Iwasawa decomposition of $\mathfrak{g}$.

Example. Let $G = SL(n,R)$, $K = SO(n)$. Then we may take A_p to be the group of real diagonal matrices with *positive* elements (and determinant 1), and N to be the group of upper triangular matrices with 1's down the diagonal. The manifold P in the Cartan decomposition consists of the positive definite symmetric matrices with determinant 1 (because if $X \in \mathfrak{p}$ is a symmetric matrix, $\exp(X)$ is symmetric and positive definite). We have $\pi_1(SL(n,R)) = \pi_1(SO(n)) = Z$ if $n = 2$, $Z/2Z$ if $n > 2$.

3. Maximal compact subgroups

Let G be a (connected) semisimple Lie group, K_o a maximal compact subgroup of G. Then $X = G/K_o$ may be given the structure of a complete simply-connected Riemannian manifold with negative curvature. If K_1 is any compact subgroup of G, then K_1 acts on X by left translations as a group of isometries of X, and by a well-known theorem of Riemannian geometry this action has a fixed point $\dot{x} = xK_o \in X$. This means that $yxK_o = xK_o$ for all $y \in K_1$, i.e. that $x^{-1}K_1x \subset K_o$. In particular, if K_1 is a *maximal* compact subgroup of G, we have $x^{-1}K_1x = K_o$, and therefore all maximal compact subgroups of G are conjugate in G.

4. Parabolic subgroups

As in §2 let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_p + \mathfrak{n} \quad , \qquad G = KA_pN$$

be Iwasawa decompositions of $\mathfrak{g}$ and G, and assume from now on that *the centre of* G *is finite* (so that K is a maximal compact subgroup of G). Let M and M^* be respectively the centralizer and normalizer of $\mathfrak{a}_p$ (or A_p) in K. Then M^*/M acts on $\mathfrak{a}_p$, hence by transposition also on the dual space $\mathfrak{a}_p^*$, and is isomorphic to the relative Weyl group $W = W(\mathfrak{g},\mathfrak{a}_p)$. The group

$$P_o = MA_pN$$

is a closed subgroup of G (because both M and A_p normalize N) and is the normalizer of N in G. P_o and its conjugates in G are the *minimal parabolic subgroups* of G.

Warning: the group M (and therefore also P_o) *need not be connected*. (However, it has at most finitely many connected components.) In any event, the identity component M^o of M is the subgroup of G corresponding to the Lie algebra $\mathfrak{m}$, the centralizer of $\mathfrak{a}_p$ in $\mathfrak{k}$. Hence the Lie algebra of P_o is

$$\mathfrak{m} + \mathfrak{a}_p + \mathfrak{n} = \mathfrak{l} + \sum_{\lambda>0} \mathfrak{g}^{\lambda}$$

in the notation of II, §9.

Example. Let $G = SL(n,R)$ and let K, A_p, N be the subgroups defined in the Example in §2. Then M consists of the diagonal matrices in which each element is ± 1 (and determinant equal to 1), hence is a finite group of order 2^{n-1}; MA_p is therefore the group of *all* diagonal matrices in G, and $P_o = MA_pN$ the group of upper triangular matrices in G. So P_o has 2^{n-1} connected components, corresponding to the various choices of sign for the diagonal elements.

If $x \in M^*$, the double coset P_oxP_o depends only on the coset xM, that is to say on the image w of x in $M^*/M = W$, so that we may write $P_oxP_o = P_owP_o$ unambiguously. We have then

$$G = \bigcup_{w \in W} P_owP_o \qquad \text{(disjoint union)}.$$

This is the *Bruhat decomposition* of G. The group W has a unique element w_1 which transforms each positive root $\lambda \in S^+$ into a negative root, and $P_ow_1P_o$ is a dense open submanifold of G, whose complement has zero Haar measure. All the double cosets $P_o w P_o$ are locally closed submanifolds of G.

The pair of subgroups (P_o, M^*) is a *BN-pair* or *Tits system* in G. Abstractly, a Tits system in a group G consists of a pair of subgroups B, N which together generate G and satisfy certain axioms which we shall not reproduce here. The group $H = B \cap N$ is normal in N, and $W = N/H$ is called the *Weyl group*; it has a distinguished set Δ of involutory generators. For each $x \in N$, the double coset BxB depends only on the image $w = xH$ of x in W, and is denoted by BwB. It is then a consequence of the axioms of a Tits system that G has a Bruhat decomposition

$$G = \bigcup_{w \in W} BwB \qquad \text{(disjoint union).}$$

For each subset E of Δ let W_E be the subgroup of W generated by E. Then $P_E = BW_E B$ is a subgroup of G, and the mapping $E \to P_E$ is an inclusion-preserving bijection of the set of subsets of Δ onto the set of subgroups of G which contain B. Each group P_E is its own normalizer, and no two of them are conjugate in G. In particular, $P_\emptyset = B$ and $P_\Delta = G$. Generalizing the Bruhat decomposition we have

$$G = \bigcup_{w \in W_E \backslash W / W_F} P_E w P_F \qquad \text{(disjoint union)}$$

for any subsets E, F of Δ.

In the present situation, B is the minimal parabolic subgroup P_o, and N is the normalizer M^* of $A_\mathfrak{p}$ in K. The group $H = B \cap N$ is $M^* \cap P_o = M$, and $W = M^*/M$ is the relative Weyl group of G. The subgroups P_E and their conjugates in G are called *parabolic* subgroups. They form $2^{\ell'}$ conjugacy classes, where $\ell' = \mathrm{card}(\Delta) = \dim \mathfrak{a}_\mathfrak{p}$ is the relative rank of $\mathfrak{g}$.

5. The Langlands decomposition of a parabolic subgroup

We retain the notation and assumptions of §3. As in II, §10, let $S = S(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ be the relative root system of $\mathfrak{g}$, and let Δ be the basis of S determined by the Iwasawa decomposition (in which $\mathfrak{n} = \sum_{\lambda \in S^+} \mathfrak{g}^\lambda$, where S^+ is the set of positive roots defined by Δ).

Let E be any subset of Δ, $\langle E \rangle$ the subsystem of S generated by E. Let $\mathfrak{g}^E$ be the Lie algebra generated by the root spaces $\mathfrak{g}^\lambda$ for $\lambda \in \langle E \rangle$. (If $E = \Delta$, then $\mathfrak{g}^E = \mathfrak{g}$.) Then $\mathfrak{g}^E$ is a semisimple subalgebra of $\mathfrak{g}$. If we put $\mathfrak{k}^E = \mathfrak{k} \cap \mathfrak{g}^E$ and $\mathfrak{p}^E = \mathfrak{p} \cap \mathfrak{g}^E$, then $\mathfrak{g}^E = \mathfrak{k}^E + \mathfrak{p}^E$ is a Cartan decomposition of $\mathfrak{g}^E$, and $\mathfrak{a}_\mathfrak{p}^E = \mathfrak{a}_\mathfrak{p} \cap \mathfrak{g}^E$ is a maximal abelian subalgebra of the vector space $\mathfrak{p}^E$.

Let $\mathfrak{a}_E$ be the orthogonal complement (with respect to the Killing form $B_\mathfrak{g}$) of $\mathfrak{a}_\mathfrak{p}^E$ in $\mathfrak{a}_\mathfrak{p}$. Equivalently, $\mathfrak{a}_E$ is the intersection of the kernels of the linear forms $\lambda \in E$. As in II, §9, $\mathrm{ad}_\mathfrak{g}(\mathfrak{a}_E)$ is a commuting set of self-adjoint endomorphisms of $\mathfrak{g}$, and so we have a root-space decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{l}_E + \sum_{\mu \in S_E} \mathfrak{g}^\mu \\ &= \mathfrak{l}_E + \mathfrak{n}_E + \bar{\mathfrak{n}}_E \end{aligned}$$

where $\mathfrak{l}_E$ is the centralizer of $\mathfrak{a}_E$ in $\mathfrak{g}$, and $S_E = S - \langle E \rangle$, and $\mathfrak{n}_E = \sum_{\mu \in S_E^+} \mathfrak{g}^\mu$, $\bar{\mathfrak{n}}_E = \sum_{\mu \in S_E^+} \mathfrak{g}^{-\mu}$. Now define

$$\mathfrak{m}_E = \mathfrak{l}_E \cap \mathfrak{k} + [\mathfrak{l}_E, \mathfrak{l}_E] \cap \mathfrak{p} ;$$

then $\mathfrak{m}_E$ is a reductive subalgebra of $\mathfrak{g}$, and we have

$$\mathfrak{l}_E = \mathfrak{m}_E + \mathfrak{a}_E \qquad \text{(direct sum).}$$

We shall now pass from the Lie algebra to the group G. Let $A_\mathfrak{p}^E, A_E, L_E^o, M_E^o, N_E$ respectively be the connected Lie groups immersed in G which correspond to the Lie algebras $\mathfrak{a}_p^E, \mathfrak{a}_E, \mathfrak{l}_E, \mathfrak{m}_E, \mathfrak{n}_E$. All these subgroups are closed in G ; we have $A_p = A_p^E A_E$ (direct product), and $L_E^o = M_E^o A_E$.

Let L_E be the centralizer of $\mathfrak{a}_E$ (or A_E) in G, so that L_E^o is the identity component of L_E . Put $M_E(K) = L_E \cap K$, the centralizer of A_E in K , and let $M_E = M_E(K)M_E^o$, so that M_E^o is the identity component of M_E . We have $L_E = M_E A_E$, and the groups L_E, M_E and

$$P_E = M_E A_E N_E = L_E N_E$$

are closed subgroups of G , and P_E contains the minimal parabolic subgroup P_o $(= P_\emptyset)$. So the groups P_E , for all subsets E of Δ , are the parabolic subgroups of G which contain P_o . The Lie algebra of P_E is $\mathfrak{l}_E + \mathfrak{n}_E = \mathfrak{p}_E$.

Example. Let us illustrate all this by reference to the group $SL(n,R)$. As before (§3, Ex.) we take as minimal parabolic P_o the group of upper triangular matrices with determinant 1, and we take $K = SO(n)$. The relative root system S is the same as the absolute root system R , and is of type A_{n-1} . Each subset E of the basis Δ may be described by a subset $\{m_1,\dots,m_r\}$ of $\{1,2,\dots,n-1\}$, where $m_1 < m_2 < \dots < m_r$, or equivalently by the sequence

$$(n_1,n_2,\dots,n_{r+1}) = (m_1, m_2-m_1,\dots,m_r-m_{r-1}, n-m_r)$$

of positive integers whose sum is n . Correspondingly we write each matrix $X \in SL(n,R)$ or $\mathfrak{sl}(n,R)$ in block form: $X = (X_{ij})$ where X_{ij} has n_i rows and n_j columns $(1 \le i,j \le r+1)$. Then $\mathfrak{a}_E$ (resp. A_E) consists of all block diagonal matrices $(X_{ij} = 0$ if $i \neq j)$ with $X_{ii} = x_i I_{n_i}$ $(1 \le i \le r+1)$ and $\Sigma n_i x_i = 0$ (resp. each $x_i > 0$ and $\Pi x_i^{n_i} = 1$) . The centralizer $\mathfrak{l}_E$ of $\mathfrak{a}_E$ consists of all block diagonal matrices X with trace 0 , and $\mathfrak{m}_E$ consists of block diagonal matrices X with trace $X_{ii} = 0$ $(1 \le i \le r+1)$, i.e. $\mathfrak{m}_E \cong \prod_i \mathfrak{sl}(n_i,R)$. The corresponding groups are

$$L_E^o = SL(n,R) \cap (\prod_i GL^+(n_i,R))$$

$$L_E = SL(n,R) \cap (\prod_i GL(n_i,R))$$

(so that $(L_E : L_E^o) = 2^r$)

$$M_E(K) = L_E \cap K = SL(n,R) \cap \prod_i O(n_i)$$

$$M_E^o = \prod_i SL(n_i,R)$$

$$M_E = \{\operatorname{diag}(X_1,\dots,X_{r+1}),\ X_i \in GL(n_i,R),\ \det X_i = \pm 1\}$$

$$N_E = \{X = (X_{ij}) : X_{ij} = 0 \text{ if } i > j,\ X_{ii} = I_{n_i}\}$$

$$P_E = \{X \in (X_{ij}) : X_{ij} = 0 \text{ if } i > j,\ \det X = 1\}$$

So P_E consists of the matrices in $SL(n,R)$ which are upper triangular in the block form determined by $(n_1,n_2,\dots,n_{r+1})$.

The subgroup N_E of the parabolic group P_E can be characterized intrinsically: it is the *unipotent radical* of P_E , i.e. the largest connected normal subgroup of P_E whose elements are unipotent, and its Lie algebra $\mathfrak{n}_E$ is the largest ideal of $\mathfrak{p}_E$ whose elements are nilpotent. The subgroup L_E is a *Levi subgroup* of P_E , that is to say a reductive closed subgroup L of P_E such that $P_E = L.N_E$ (semidirect product). Hence if P is any parabolic subgroup of G , N_P its unipotent radical, we have

$$P = L.N_P \qquad \text{(semidirect product)} \qquad (1)$$

where L is a Levi subgroup of P (*Levi decomposition* of P). The Levi component L is not unique, but if L' is another then we have $L' = xLx^{-1}$ for some $x \in N_P$.

The subgroup A_E above can be characterized as the largest connected split abelian subgroup of the centre of the Levi subgroup L_E (an abelian subgroup A of G is *split* if for each $x \in A$, $\mathrm{Ad}_G(x)$ is diagonalizable over R). The group M_E can be described as the intersection of the kernels of all continuous homomorphisms $\chi : L_E \to R$. Hence, if P is any parabolic subgroup of G and L a Levi subgroup of P we have $L = MA$, $M \cap A = \{1\}$, and hence by (1)

$$P = MAN_P \tag{2}$$

where $M = \cap \mathrm{Ker}(\chi : L \to R)$ and A is the largest connected split abelian subgroup of the centre of L; moreover the product mapping of $M \times A \times N_P$ onto P is a diffeomorphism. This is the *Langlands decomposition* of P. The group A is called a *split component* of P; it is unique up to conjugation by elements of N_P. The dimension of A is called the *parabolic rank* of P. (Thus the parabolic rank of P_E is $\mathrm{card}(\Delta - E)$.)

Two parabolic subgroups P,P' are said to be *associated* if P and $xP'x^{-1}$ have a common Levi subgroup, for some $x \in G$. Clearly conjugate parabolics are associated, but the converse is false. For example, in $G = SL(n,R)$, the parabolic subgroups $P_E, P_{E'}$ are associated if and only if the sequences (n_i) and (n_i') determined respectively by E and E' are permutations of each other. Hence the number of classes of associated parabolic subgroups in $SL(n,R)$ is equal to the number of partitions of n (whereas the number of conjugacy classes of parabolic subgroups is 2^{n-1}).

6. Cartan subgroups

As before, let G be a connected semisimple Lie group with finite centre, $\mathfrak{g}$ the Lie algebra of G. A *Cartan subgroup* of G is the centralizer B in G of a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$: $B = Z_G(\mathfrak{b})$. It is a closed subgroup of G, but is not necessarily connected. Its identity component B^o is the closed subgroup $\exp_G(\mathfrak{b})$ with Lie algebra $\mathfrak{b}$, and the group of components B/B^o is finite.

Example. Let $G = SL(2,R)$ and let $\mathfrak{a}$, $\mathfrak{b}$ be the Cartan subalgebras of $\mathfrak{g} = \mathfrak{sl}(2,R)$ generated by $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ respectively (II, §11). The corresponding Cartan subgroups of G are A, consisting of all diagonal matrices $\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}$ (so that $A \cong R^*$ and therefore has two components) and $B = SO(2)$ (which is connected).

Another warning: Cartan subgroups need not be abelian. (If $\mathfrak{a}$ is the Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(3,R)$ consisting of the diagonal matrices, and G is the simply-connected double covering group of $SL(3,R)$, then the centralizer A of $\mathfrak{a}$ in G is not abelian.) However, if G admits a faithful finite-dimensional representation, then all Cartan subgroups of G are abelian; so that in any case the Cartan subgroups of G are abelian modulo the centre of G.

Moreover, if $\mathfrak{b}$ is a maximally compact (or fundamental) Cartan subalgebra of $\mathfrak{g}$, then $B = Z_G(\mathfrak{b})$ is both connected and abelian.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decompositon of $\mathfrak{g}$, let θ be the associated Cartan involution and let $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} + \mathfrak{b}_{\mathfrak{p}}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let B be the centralizer of $\mathfrak{b}$ in G, let $B_{\mathfrak{p}} = \exp(\mathfrak{b}_{\mathfrak{p}})$, and $B_K = B \cap K$, where K (§2) is the compact subgroup of G with $\mathfrak{k}$ as Lie algebra. $B_{\mathfrak{p}}$ is a vector group (the *vector part* of B) and B_K is a (not necessarily connected) compact group (the *compact part* of B), and we have $B = B_K . B_{\mathfrak{p}}$. The identity component B_K^o of B_K is the subgroup of G corresponding to $\mathfrak{b}_{\mathfrak{k}}$.

Examples

1. If $G = SL(n,R)$, $\mathfrak{g} = \mathfrak{sl}(n,R)$, there are up to conjugacy in $\mathfrak{g}$ $[\frac{1}{2}n] + 1$ distinct Cartan subalgebras $\mathfrak{b}_j$ $(0 \le j \le [\frac{1}{2}n])$, where the elements of $\mathfrak{b}_j$ are diagonal sums of j 2×2 matrices $\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ and a diagonal matrix of size $n-2j$ (II, §11, Examples). The centralizer B_j of $\mathfrak{b}_j$ in G consists of the matrices of the same description and determinant equal to 1 . Since the group of nonzero matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is isomorphic to the multiplicative group C^* , it follows that B_j is isomorphic to $(C^*)^j \times (R^*)^{n-2j-1}$ (unless $n = 2j$), hence has 2^{n-2j-1} connected components. If $n = 2j$, B_j is isomorphic to $(C^*)^{j-1} \times T \cong T^k \times R^{k-1}$, where T is the circle group.

2. If $G = SL(n,H)$, $\mathfrak{g} = \mathfrak{sl}(n,H)$, there is up to conjugacy in $\mathfrak{g}$ only one Cartan subalgebra $\mathfrak{a}$, consisting of the complex diagonal matrices $X \in \mathfrak{g}$ such that $\mathrm{Re}(\mathrm{trace}\ X) = 0$ (II, §10, Examples). The corresponding Cartan subgroup A of G consists of the complex diagonal matrices X such that $|\det X| = 1$, hence is isomorphic to $(C^*)^{n-1} \times T \cong T^n \times R^{n-1}$.

3. If $G = SU(p,q)$, the group of linear transformations of C^n with determinant 1 which leave invariant the Hermitian form

$$z_1\bar{z}_1 + \dots + z_p\bar{z}_p - z_{p+1}\bar{z}_{p+1} - \dots \ z_n\bar{z}_n ,$$

then $\mathfrak{g} = \mathfrak{su}(p,q)$ has $q+1$ conjugacy classes of Cartan subalgebras (II, §11, Examples), represented by $\mathfrak{b}^{(j)}$ $(0 \le j \le q)$ (loc.cit.). The centralizer $B^{(j)}$ of $\mathfrak{b}^{(j)}$ in G consists of the matrices

$$\begin{pmatrix} e^{iX}\,\mathrm{ch}\,Y & 0 & e^{iX}\,\mathrm{sh}\,Y \\ 0 & e^{iZ} & 0 \\ e^{iX}\,\mathrm{sh}\,Y & 0 & e^{iX}\,\mathrm{ch}\,Y \end{pmatrix}$$

where X,Y,Z are real diagonal matrices of sizes $q-j$, $q-j$ and $p-q+2j$ respectively, and $2\ \mathrm{trace}\ X + \mathrm{trace}\ Z = 0$; $B^{(j)}$ is connected and abelian, isomorphic to $T^{p+j-1} \times R^{q-j}$.

7. The regular set

Let G be a connected semisimple Lie group with finite centre, $\mathfrak{g}$ its Lie algebra. As in Chapter I, §4, let $\mathfrak{g}'$ (resp. G') denote the set of regular elements of $\mathfrak{g}$ (resp. G) . For any Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, let $\mathfrak{b}' = \mathfrak{g}' \cap \mathfrak{b}$, and define

$$\mathfrak{g}_{\mathfrak{b}'} = \bigcup_{x \in \mathrm{Int}(\mathfrak{g})} x(\mathfrak{b}') = \bigcup_{x \in G} \mathrm{Ad}(x)(\mathfrak{b}') .$$

Likewise, if B is a Cartan subgroup of G , let $B' = G' \cap B$, and define

$$G_{B'} = \bigcup_{x \in G} xB'x^{-1} .$$

Now let $\mathfrak{b}_i$ $(1 \le i \le r)$ be a set of representatives of the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$, and let B_i be the centralizer of $\mathfrak{b}_i$ in G , so that every Cartan subgroup of G is conjugate to exactly one of the B_i . Then we have

$$\mathfrak{g}' = \bigcup_{i=1}^{r} \mathfrak{g}_{\mathfrak{b}_i'}$$

and

$$G' = \bigcup_{i=1}^{r} G_{B_i'} .$$

Since each $\mathfrak{b}_i'$ (resp. B_i') has only a finite number of connected components, it follows that the number of components of $\mathfrak{g}'$ (resp. G') is finite.

Let again $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$, and let B, B^* be respectively the centralizer and normalizer of $\mathfrak{b}$ in G. The group $B^*/B = W(G,B)$ is finite. It acts on G/B by right multiplication: if $w = nB \in W(G,B)$ and $\dot{x} = xB \in G/B$, then $\dot{x}w = xBnB = xnB$. Also $W(G,B)$ acts on $\mathfrak{b}'$ by the rule $w.H = Ad_G(n).H$. Hence $W(G,B)$ acts on $(G/B) \times \mathfrak{b}'$: $w(\dot{x},H) = (\dot{x}w^{-1}, wH)$. Let

$$\phi : (G/B) \times \mathfrak{b}' \to {}^{\mathfrak{g}}\mathfrak{b}'$$

be the mapping $(\dot{x},H) \to Ad_G(x)H$. Then ϕ is an everywhere regular covering map of degree $|W(G,B)|$.

For the global analogue of this result we must replace the Cartan subgroup B by its *centre* B_o, since B might not be abelian. Define $W(G,B_o) = B^*/B_o$, which is still a finite group. This group acts (on the right) on G/B_o and by conjugation on B'. Let

$$\psi : (G/B_o) \times B' \to {}^{G}B'$$

be the mapping $(\dot{x},b) \to xbx^{-1}$. Then ψ is an everywhere regular covering map of degree $|W(G,B_o)|$.

These results enable integration over $\mathfrak{g}$ (resp. G) to be reduced to integration over $(G/B_i) \times \mathfrak{b}_i$ (resp. $(G/B_{io}) \times B_i$), $i = 1,\ldots,r$, on the lines of Weyl's integration formula for compact Lie groups.

8. Complex Lie groups

In this section we shall briefly review the structure theory of *complex* semisimple Lie groups, which is a much simpler business than the real theory: in particular, the phenomena of disconnectedness (of Cartan subgroups and parabolic subgroups) do not arise in the complex case.

A *complex Lie group* is a complex-analytic manifold G which is a group, the group operations being holomorphic mappings. Semisimplicity, Cartan subgroups, parabolic subgroups etc. are defined exactly as in the real case.

Let G be a complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, H the centralizer of $\mathfrak{h}$ in G (the Cartan subgroup of G corresponding to $\mathfrak{h}$). Then $H = \exp_G(\mathfrak{h})$ and is a closed complex-analytic subgroup of G, isomorphic to $(C^*)^\ell$ where $\ell = \dim_C \mathfrak{h}$ is the (complex) rank of $\mathfrak{g}$ (or of G). All Cartan subgroups of G are conjugate to H (because all Cartan subalgebras of $\mathfrak{g}$ are conjugate to $\mathfrak{h}$).

As in Chapter II, §3 let $R \subset \mathfrak{h}_R^*$ be the root system of $(\mathfrak{g},\mathfrak{h})$. We define three lattices L_0, L_1 and L_G in $\mathfrak{h}_R$, as follows. L_1 is the lattice of all $X \in \mathfrak{h}_R$ such that $\alpha(X) \in Z$ for all roots $\alpha \in R$, i.e. L_1 is the dual of the lattice in $\mathfrak{h}_R^*$ spanned by the roots. L_0 is the lattice in $\mathfrak{h}_R$ spanned by the vectors $H_{\alpha^\vee}$ corresponding to the coroots $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$, $\alpha \in R$. Since $\beta(H_{\alpha^\vee}) = \langle\alpha^\vee,\beta\rangle \in Z$ for any two roots $\alpha,\beta \in R$, it follows that L_0 is a sublattice of L_1. Moreover the quotient L_1/L_0 is a *finite* group. Both L_0 and L_1 depend only on the root system R. Finally, the lattice L_G is the kernel of the homomorphism $e : \mathfrak{h} \to H$ defined by $e(X) = \exp_G(2\pi i X)$; e is surjective and therefore induces an isomorphism $\mathfrak{h}/L \cong H$. The lattice L_G lies between L_0 and L_1; also

(1) The homomorphism $e : \mathfrak{h} \to H$ defines an isomorphism of L_1/L_G onto the centre of G.

(2) The canonical mapping $\pi_1(H) \to \pi_1(G)$ is surjective and defines an isomorphism of L_G/L_0 onto $\pi_1(G)$.

It follows that $L_G = L_0$ if and only if G is simply connected, and that $L_G = L_1$ if and only if G is adjoint (i.e. $G = \mathrm{Ad}(G)$). For each lattice L lying between L_0 and L_1 there exists a connected complex semisimple Lie group G with Lie algebra $\mathfrak{g}$ such that $L_G = L$, and G is unique (up to isomorphism).

Example. Let $G = SL(n,C)$, and take H to be the diagonal subgroup of G . Then $\mathfrak{h}_R$ consists of the real diagonal matrices with trace 0 ; L_0 consists of the diagonal matrices with trace 0 and *integer* elements (so that $L_0 \cong Z^{n-1}$) and L_1 consists of the diagonal matrices $\mathrm{diag}(a_1,\dots,a_n)$ with $\Sigma a_i = 0$ and $a_i - a_j \in Z$ for all i,j . It follows that L_1/L_0 is cyclic of order n ; we have $L_G = L_0$, in agreement with the fact that $SL(n,C)$ is simply-connected. Hence the almost simple connected complex Lie groups with Lie algebra $\mathfrak{sl}(n,C)$ are in one-one correspondence with the subgroups of a cyclic group of order n , i.e. with the divisors of n .

Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}^{\alpha}$$

be the root-space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (II, §3). Let R^+ be the set of positive roots of R relative to a chosen basis, and let

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in R^+} \mathfrak{g}^{\alpha} \quad .$$

Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, called a *Borel subalgebra*. It is a maximal solvable subalgebra of $\mathfrak{g}$, and every solvable subalgebra of $\mathfrak{g}$ is conjugate to a subalgebra of $\mathfrak{b}$.

Let B be the immersed subgroup of G corresponding to $\mathfrak{b}$. Then B is a closed complex-analytic subgroup of G , called a *Borel subgroup* of G . It is a maximal connected closed solvable subgroup of G , and every connected solvable subgroup of G is conjugate to a subgroup of B .

In the terminology of §4, the Borel subgroups are the minimal parabolic subgroups of G . The parabolic subgroups of G are therefore the subgroups of G which contain a Borel subgroup; they are closed, connected complex-analytic subgroups of G . Moreover, a closed complex-analytic subgroup Q of G is a parabolic subgroup if and only if the homogeneous space G/Q is compact.

Let K be a maximal compact subgroup of G , B a Borel subgroup. Then $T = B \cap K$ is a maximal torus (i.e. Cartan subgroup) of the compact (real) Lie group K . Since $G = KB$ (Iwasawa decomposition) we have

$$G/B = KB/B \cong K/B \cap K = K/T \quad .$$

More generally, if P is a parabolic subgroup of G , then $K_P = P \cap K$ is a subgroup of K containing a maximal torus, and $G/P \cong K/K_P$.

9. Lie groups and algebraic groups

Let $\mathfrak{g}$ be a real semisimple Lie algebra. The adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ maps $\mathfrak{g}$ isomorphically onto a subalgebra of $\mathfrak{gl}(\mathfrak{g}) \cong \mathfrak{gl}(n,R)$ where $n = \dim \mathfrak{g}$. Thus $\mathfrak{g}$ can be realized as a Lie algebra of matrices.

On the other hand, if G is a real semisimple Lie group, it is not necessarily the case that G can be realized as a group of matrices - or, equivalently, that G has a faithful finite-dimensional representation - even if the centre of G is finite.

A semisimple Lie group G is said to be *linear* if G has a faithful finite-dimensional representation, i.e. if there exists a smooth injective homomorphism $i : G \to GL(n,C)$ for some n . It follows then that $i(G)$ is a *closed* subgroup of $GL(n,C)$. An equivalent condition is that G is *algebraic*, i.e. isomorphic to the identity component (with respect to the usual topology) of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$.

Let $\tilde{G}$ and $\tilde{G}_C$ be the simply-connected Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{g}_c$ respectively, and let $\phi : \tilde{G} \to \tilde{G}_C$ be the homomorphism corresponding to the injection $X \to X \otimes 1$ of $\mathfrak{g}$ into $\mathfrak{g}_c$. This homomorphism ϕ is not necessarily injective; it maps the centre of $\tilde{G}$ into the centre of $\tilde{G}_C$, and its kernel D_o is a subgroup of the (discrete) centre of $\tilde{G}$. The group $G_o = \tilde{G}/D_o$ is the 'largest algebraic quotient' of $\tilde{G}$.

If G is any connected Lie group with $\mathfrak{g}$ as Lie algebra, let $N(G)$ denote the intersection of the kernels of the finite-dimensional representations of G . Then (1) there exists a representation of G whose kernel is exactly $N(G)$, and (2) $N(G)$ is the image of $D_o = \ker(\phi)$ under the covering map $\tilde{G} \to G$.

Example. If $G = SL(n,\mathbb{R})$, then $\tilde{G}_C = SL(n,\mathbb{C})$ and $\phi : \tilde{G} \to \tilde{G}_C$ is the composition of the covering map $p : \tilde{G} \to G$ with the embedding $SL(n,\mathbb{R}) \to SL(n,\mathbb{C})$. Consequently no proper covering group of $SL(n,\mathbb{R})$ is algebraic.

Finally, if G is compact and semisimple (which means that the centre of G is finite) then $\tilde{G}$ is compact, ϕ is injective, and ϕ maps the centre of $\tilde{G}$ isomorphically onto the centre of $\tilde{G}_C$. Hence every compact semisimple group is algebraic, and

$$\tilde{G}/D \longleftrightarrow \tilde{G}_C/\phi(D)$$

(where D is a subgroup of the (finite) centre of $\tilde{G}$) sets up a one-one correspondence (up to isomorphism) between compact semisimple groups and complex semisimple groups.

References

Chapter I

§1 Chevalley Ch.III, Helgason Ch.I

§2-6 Bourbaki Ch.III, Chevalley Ch.IV, Dieudonné Ch.XIX, Helgason Ch.II

§7 Bourbaki Ch.III, Dieudonné Chs.XVI,XIX

§8 Dieudonné Chs.XVI,XIX

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§1 Bourbaki Ch.I, Jacobson Chs.I-III, Serre Chs.I,II

§2 Bourbaki Ch.VII, Humphreys Ch.IV, Jacobson Ch.III, Serre Ch.III

§3 Bourbaki Ch.VIII, Dieudonné Ch.XXI, Helgason Ch.III, Humphreys Chs.II,IV, Jacobson Ch.IV, Serre Ch.VI

§4 Bourbaki Ch.VI, Dieudonné Ch.XXI, Humphreys Ch.III, Jaconson Ch.IV, Serre Ch.V

§5 Bourbaki Ch.VI, Humphreys Ch.V, Jacobson Ch.IV, Serre Chs.V,VI

§6 Dieudonné Ch.XXI, Helgason Ch.III

§7 Helgason Ch.IX, Tits

§8 Hegason Ch.III, Dieudonné Ch.XXI

§9 Dieudonne Ch.XXI, Helgason Ch.VI, Warner Ch.I

§10 Helgason Ch.VI, Warner Ch.I

§11 Warner Ch.I

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§1 Bourbaki Ch.III, Dieudonné Ch.XXI, Helgason Ch.II

§2 Dieudonné Ch.XXI, Helgason Ch.VI

§3 Helgason Ch.VI

§4-7 Warner Ch.I

§8 Helgason Ch.VII, Tits, Warner Ch.I

§9 Tits, Warner Ch.II

Bibliography

N.Bourbaki, Groupes de Lie et Algèbres de Lie, Chapters I-VIII (Hermann, 1960-1975)