Tutte polynomial

For this topic, we allow loops and multiple edges in graphs.

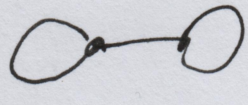
The Tutte polynomial is a two variable poly attached to a graph G. Like the chromatic poly, it's defined via a "deletion-contraction" recursion.

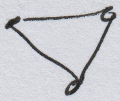
<u>Def</u> The Tutte polynomial $T_G(x,y)$ is defined as follows:

1. If G consists entirely of loops and bridges, then $T_G(x,y) = x^i y^j$, where i is the number of bridges and j is the number of loops.
2. If e is an edge of G that is neither a loop nor a bridge, then

$$T_G(x,y) = T_{G-e}(x,y) + T_{G/e}(x,y)$$

<u>Some examples</u>

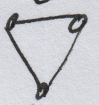
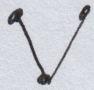
1. $G =$  $\Rightarrow T_G = xy^2$
2. If G is a tree of order n, then $T_G = x^{n-1}$ (because all edges are bridges)

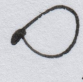
3. $G = K_3 =$ 

We illustrate the procedure graphically

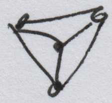
e; not a bridge, not a loop

don't throw away multiple edges

$T($  $) = T($  $) + T($  $)$

$= x^2 + T($  $) + T($  $)$

$= x^2 + x + y.$

4. $G = K_4 =$ 

This takes some effort. The final answer is

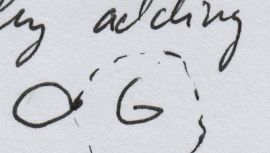
$$x^3 + 3x^2 + 4xy + 2x + y^3 + 3y^2 + 2y.$$

We can do some preliminary computations first.

(a) $T($  $) = x + y + y^2 + \dots + y^{n-1}$

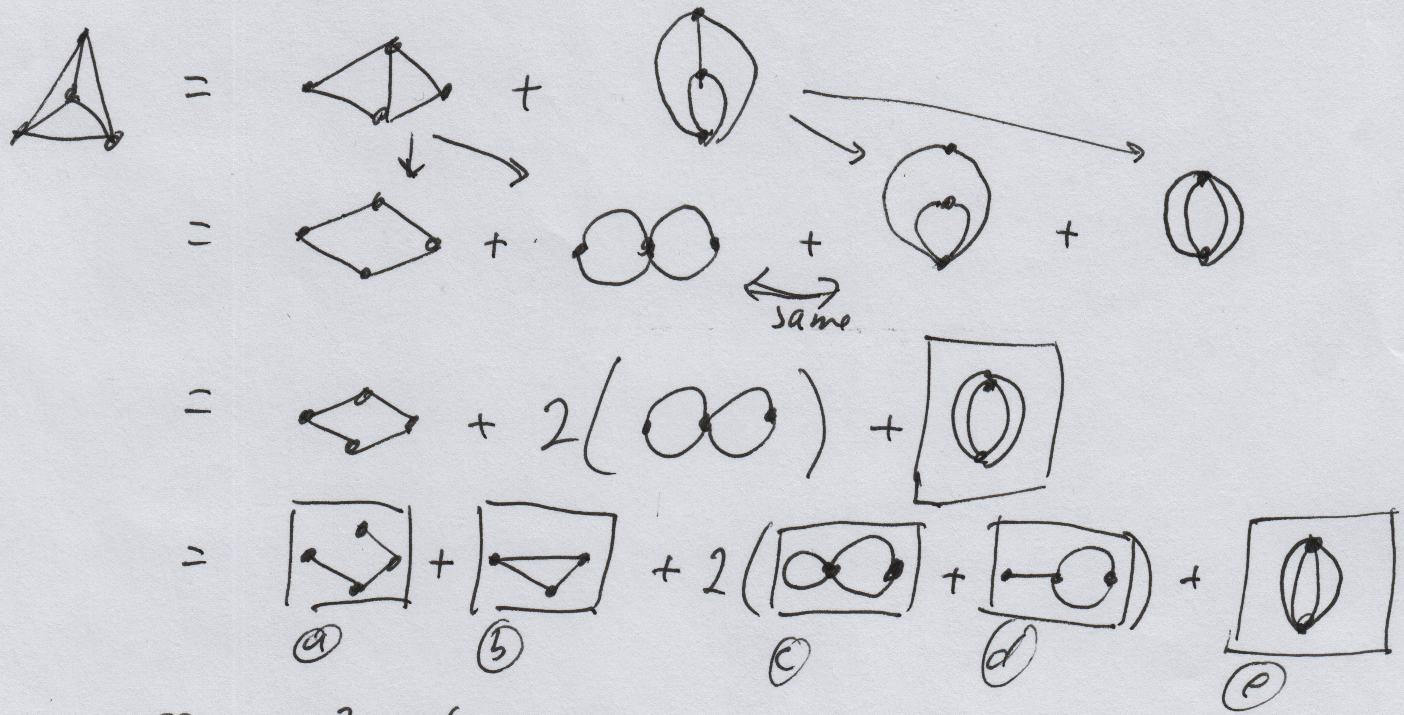
(n-fold multiple edge)

(b) If G' is obtained from G by adding a loop, then $T_{G'} = y\,T_G$:



(c) Similarly, if G' is obtained from G by poking off an edge, then $T_{G'} = x\,T_G$:

Now we can try K_4. (I'll omit the $T(\cdot)$ to simplify the notation, and will box graphs when we know their polynomials.)



ⓐ $= x^3$ (tree)

ⓑ $= x^2 + x + y$ (K_3)

2ⓒ $= 2xy + 2y^2$ (loop on O)

2ⓓ $= 2x^2 + 2xy$ (— on O)

ⓔ $= x + y + y^2 + y^3$ (4-fold multiple edge)

$x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$ ✓

Connection between Tutte and the chromatic polynomial

Let n be the order of G and $k(G)$ the number of connected components of G. Then

$$C_G(q) = (-1)^{n-k(G)} q^{k(G)} T_G(1-q, 0)$$

(we're using q instead of k for the variable)

Examples

① If G is a tree of order n, then $T_G(x,y) = x^{n-1}$. $k(G) = 1$, so we get

$$(-1)^{n-1} q^{1} (1-q)^{n-1} = q(q-1)^{n-1} \checkmark$$

② If $G = K_3$, then $T_G = x^2 + x + y$. $n = 3$, $k(G) = 1$. So we get

$$(-1)^2 q^{1} \left((q-1)^2 + (1-q) + 0\right)$$

$$= q^3 - 3q^2 + 2q = q(q-1)(q-2) \checkmark$$