MATH 455 PROBLEM SET HINTS

PROBLEM SET II

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.2.1.

(1) (Check me!) The radius is 3, the diameter is 6, and the center is the central vertex.

(2) I'll just do a few. To get these just do a bunch of examples and guess the connection between r, d and k. (r is the radius, d is the diameter.)
   (1) P_{2k}: $r = k$, $d = 2k - 1$.
   (2) C_{2k}: $r = d = k$.
   (3) K_n: $r = d = 1$.

(5) The key is the triangle inequality: if x is any vertex, then $d(x, u) \leq d(x, v) + 1$ and $d(x, v) \leq d(x, u) + 1$. So suppose $ecc(u) = a$ and $ecc(v) = a + k$, $k \geq 2$. Let x be a vertex with $d(x, v) = a + k$. Then $d(x, v) \leq d(x, u) + 1$, but $d(x, u)$ must be at most a, since $ecc(u) = a$. This gives $a + k \leq a + 1$, a contradiction.

§1.2.2.

(1) Simply write down the matrices after choosing labellings for the vertices as they have indicated (their choices make the answers very symmetric).

(3) Any 2-step walk from a vertex v to itself is constructed by passing from v to a neighbor and back.

(4) (a) A triangle is a three step walk from a vertex to itself, so is related to the diagonal of A^3. But if we have the walk $ijki$ we also have the walk $ikji$, so we have to divide by 2. (b) By (a) 1/2 the trace of A^3 is the number of triangles where we have distinguished one vertex. Since any triangle has three vertices, we have to divide this by three to get the actual number of triangles, so we get 1/6 of the trace of A^3.

(5) The answer is 0. We want to count the number of 2009-step walks from vertex 1 to 5. (This is not easy to do explicitly, except that it happens to be zero. For instance, the number of 2009-step walks

from vertex 1 to 4 is about 10^{604}.) Color the vertices red, black depending on whether or not their labels are even or odd. I claim there are no walks of an odd number of steps between two same colored vertices and no walks of an even number of steps between two different colored vertices (you can prove these by induction). Since 1 and 5 have the same color the corresponding entry of A^{odd} must be zero.

§1.3.1.

(1) The best way to do this is to start from the 6 trees of order 6 and try adding edges. There are 11 trees of order 7.

(3) Use induction. Call the two colors red and black. Clearly trees of order ≤ 3 are bipartite. Now take a tree T of order $n \geq 4$ and delete an edge such that the resulting two trees T', T'' have orders $< n$ and ≥ 2. They are bipartite by assumption. If the vertices of the deleted edge have different colors, then we can restore the edge and T is bipartite. Otherwise, interchange the colors in T' and then rejoin to construct T.

§1.3.2.

(1) By Theorem 1.11 a forest of order n with k components has $n - k$ edges. So to realize these examples, we need to be able to choose n and k as indicated to get the right number of edges. Only b, d, and e are possible.

(2) Use the fact that there are an odd number of vertices and the sum of the degrees of the vertices is twice the number of edges.

(5) If there is more than one path between u and v, then there must be a cycle.

(8) If v is a nonleaf then its degree must be at least two. let v', v'' be two distinct neighbors of v. Now explain why there must be no path from v' to v'' after v is removed.

(10) Method 1: delete an edge e to break T into two smaller trees where the formula applies, and then consider the different possibilities when e is redrawn. Method 2: Call a vertex *internal* if it is not a leaf. The formula can be extended so that the sum is taken over all internal vertices (why?). Now we argue by induction over the number of internal vertices. The key point is that for some $d \geq 2$ we can find an internal vertex of degree d with $d-1$ leaves. Deleting these leaves reduces the number of internal vertices, so the formula holds. Now add the leaves back in, and what happens to the formula?

§1.3.3.

(2) Argue that if it has more than one, you can use the two spanning trees to make a cycle.

(4) If e is not in some spanning tree T, then adding e to T makes a cycle in G, and thus e couldn't have been a bridge.

(5) In both cases the spanning tree constructed by the algorithm is unique. For the graph on the right it's clear, since all the weights are distinct (so there are no choices in the algorithm). For the graph on the left, there are choices made in the algorithm, but they all get made as the spanning tree is built.

(6) Follow the description and see what edges are added in what order.