MATH 455 PROBLEM SET HINTS

PROBLEM SET I

These are (usually) not complete solutions for the problems, but are intended to give you the basic ideas needed for a solution. If the basics of a problem are covered in class, either through working it out or doing a similar example, then we omit it here. Complete solutions typically involve more writing than is given here.

§1.1.1.

(1) Draw K_{10} with the vertices in a circle and then erase the diagonals that go directly across.
(2) Make a vertex for each person, and draw $A \to B$ is B is on A's list.
(3) Draw the graph with the indicated edges, and put a double edge if $*$ appears.

§1.1.2.

(1) The complete graph K_n has $n(n-1)/2$ edges (n choose 2).
(3) (a) Let N_k be the number of paths with k steps that start at c and walk away. Then we want $\sum_{k=0}^{4} N_k$. Think of counting sequences of distinct vertices. For instance, to get a walk of length 4, we need to write something like $cxyzw$, where x, y, z, w are some ordering of $\{a, b, d, e\}$. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such choices, so $N_4 = 24$. Similarly we find $N_0 = 1$, $N_1 = 4$, $N_2 = 12$, and $N_3 = 24$. So the total is 65.
  (b) This is very similar.
  (c) 10.
  (d) This is the same as the maximum length circuit in K_4. This is 4.
(9) The max will when the graph has two connected components, each of which are complete graphs. A little experimentation shows that we will want one component to be K_1 and the other to be K_{n-1}.
(11) If e is part of a cycle, it cannot be a bridge (for after you delete e you can still make a walk from the endpoints of e to each other). Conversely, if e is not part of a cycle, it must be a bridge, since after deleting e there will be no path between its endpoints.

§1.1.3.

(2) The degrees of the vertices are either r_1 or r_2. So if $r_1 \neq r_2$, the graph can't possibly be regular.

(3) It is not, since $K_{4,4}$ contains no copy of K_3 as a subgraph, and K_3 is a subgraph of K_4 (cf. Thm 1.3)

(4) It is not, since it would have to contain all vertices of K_4. But the induced graph of the full set of vertices is the original graph K_4, which is not P_4.

(7) For (a) and (b), just draw them out. For (c), the order of $L(G)$ is the same as the size of G. The size of $L(G)$ is more complicated. If a vertex v has degree r, then we get a copy of K_r as a subgraph of $L(G)$ (because each of the r edges emanating from v becomes a vertex in $L(G)$, and because any pair of these vertices in $L(G)$ must be joined by a edge in $L(G)$). By ranging v over all the vertices of G we get all the edges in $L(G)$. Since K_r has $r(r-1)/2$ edges, the total number of edges in $L(G)$ is

$$\sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}.$$

You can verify this for the graphs in (a) and (b).

(10) P has no 4-cycles. (To check this, pick a vertex and label the other vertices with their distances from the initial one. If there is a 4-cycle C containing the first one, the labels on C will be 0121. Then argue that you only have to check one other vertex, by symmetry.). It is clear that R does have some 4-cycles, so we must have $P \not\simeq R$. Do the same thing you did to P to Q and you'll see that $P \simeq Q$.