MATH 455 EXAM II

This exam is worth 100 points, with each problem worth 25 points. There are problems on *both sides* of the page. Please complete Problem 1 and then *any three* of the remaining problems. Unless indicated, you must justify your answer to receive credit for a solution. Calculators are allowed on this exam.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

Let me know if you find any mistakes in the answers.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

  (a) (5 pts) The falling factorial $n^{\underline{k}}$ is defined by $n(n-1)\cdots(n-k+1)$. **Answer:** True.

  (b) (5 pts) The number of subsets of $\{1,\dots,n\}$ of order k is $\binom{n}{k}$. **Answer:** True.

  (c) (5 pts) *Vandermonde's convolution* for binomial coefficients says $\binom{n}{k} = \frac{k}{n}\binom{n-1}{k-1}$. **Answer:** False. This is not an identity for binomial coefficients. Try $n=2, k=1$.

  (d) (5 pts) The number of orderings of a set of order n is $n^{\overline{n}}$. **Answer:** False. But it would be the falling factorial $n^{\underline{n}}$.

  (e) (5 pts) The Tutte polynomial of the n-cycle C_n is $x^{n-1}+x^{n-2}+\cdots+x+y$. **Answer:** True. Done in class.

(2) A *bit string* of length n is a sequence of the form $a_1, a_2, \dots a_n$ where each a_i is either 0 or 1.

  (a) (5 pts) How many bit strings of length 8 are there? **Answer:** $2^8 = 256$.

  (b) (8 pts) How many bit strings of length 8 have *at least* two 1's (in any position)? **Answer:** It's easier to count the strings that have fewer than two 1s. There are $1+8$ of these (no 1s + exactly one 1). So the answer is $256-9=247$.

  (c) (12 pts) How many bit strings of length 8 do *not* have 6 *consecutive* 0's? **Answer:** We can use inclusion/exclusion. Let S_i be the set of strings having 6 consecutive 0s starting at position i, where $i=1,2,3$ (no other values of i make sense). Let $N_i = |S_i|$. Then $N_1 = N_2 =$

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$N_3 = 4$, $N_{12} = N_{23} = 2$, $N_{13} = 1$, and $N_{123} = 1$. So the answer is $256 - (4+4+4) + (2+2+1) - 1 = 256 - 8 = 248$.

(3) (25 pts) Let G be the graph $K_4 - e$, where e is any edge. Compute the Tutte polynomial of G. **Answer:** Use the deletion/contraction algorithm and you get $x^3 + 2x^2 + 2xy + x + y + y^2$.

(4) (25 pts) Suppose a graph G is connected and let $T_G(x, y)$ be its Tutte polynomial. Then it is known that $T_G(1, 1)$ counts the number of spanning trees in G. Verify this fact for the graph shown in Figure 1. **Answer:** The Tutte polynomial of this graph is $x^3(x^2 + x + y)$, since it's a 3-cycle with 3 extra bridges poking off (as discussed in class, attaching one of these to a graph multiplies the Tutte polynomial by x.) Evaluating at $(1, 1)$ gives 3. This graph has exactly 3 spanning trees; they are gotten by picking an edge of the 3 cycle to delete.

(5) (a) (10 pts) Show that $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$. **Answer:** The sum is $n + 3n(n-1) + n(n-1)(n-2) = n^3$.

(b) (15 pts) Use the above to give a closed formula for $1^3 + 2^3 + \cdots + n^3$. (Hint: $\binom{a}{a} + \binom{a+1}{a} + \binom{a+2}{a} + \cdots + \binom{b}{a} = \binom{b+1}{a+1}$. A *closed formula* is one not involving any summations.) **Answer:** We plug in the formula in the hint into $1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{k=1}^{n} \binom{k}{1} + 6\binom{k}{2} + 6\binom{k}{3}$ to get $\binom{n+1}{1+1} + 6\binom{n+1}{2+1} + 6\binom{n+1}{3+1} = n^2(n-1)^2/4$.

(6) (25 pts) A ballot lists ten candidates for city council, eight candidates for the school board, and five bond issues. The ballot instructs voters to choose up to four people running for city council, rank up to three candidates for the school board, and approve or reject each bond issue. How many different ballots may be cast, if partially completed (or empty) ballots are allowed? **Answer:** Let N_{cc}, N_{sb}, N_{bi} be the number of ways to complete each of the three parts. The total number of ballots will then be $N_{cc}N_{sb}N_{bi}$. For the city council, we can choose anywhere from 0 to 4 candidates; they are not ranked, instead just picked or not. So $N_{cc} = \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} = 386$. For the school board we can choose and rank anywhere from 0 to 3. So $N_{sb} = 0! \cdot \binom{8}{0} + 1! \cdot \binom{8}{1} + 2! \cdot \binom{8}{2} + 3! \cdot \binom{8}{3} = 401$. For each bond issue we can accept, reject, or ignore (we're allowed to have partially completed ballots). So there are three possible choices for each issue and five issues, giving $N_{bi} = 3^5 = 243$. The total number is 37612998.

(7) (25 pts) A political science quiz has two parts. In the first, you must present your opinion of the four most influential secretaries-general in the history of the United Nations in a ranked list. In the second, you must name ten members of the United Nations security council in any order, including at least two permanent members of the council. If there have been eight secretaries-general in U.N. history, and there are fifteen members of the U.N. security council, including the five permanent members, how many ways can you answer the

quiz, assuming you answer both parts completely? **Answer:** First part: pick 4 from 8 and rank them: $4! \cdot \binom{8}{4} = 1680$. For the second part, we can break it up by how many we've chosen from from the permanent members. There are $\binom{5}{k}$ ways to choose k of them, and then $\binom{10}{10-k}$ ways to complete the selection to 10 total. We need k to be at least 2, so we want $\sum_{k=2}^{5} \binom{5}{k}\binom{10}{10-k}$. This gives 2952. The total number of exams is then 4959360.

(8) (25 pts) On a busy evening a number of guests visit a gourmet restaurant, and everyone orders something. 140 guests order a beverage, 190 order an entree, 100 order an appetizer, 90 order a dessert, 65 order a beverage and an appetizer, 125 order a beverage and an entree, 60 order a beverage and a dessert, 85 order an entree and an appetizer, 75 order an entree and a dessert, 60 order an appetizer and a dessert, 40 order a beverage, appetizer, and dessert, 55 order a beverage, entree, and dessert, 45 order an appetizer, entree, and dessert, 35 order a beverage, entree, and appetizer, and ten order all four types of items. How many guests visited the restaurant that evening? **Answer:** Inclusion/Exclusion. Let N be the total number of guests and N_0 the number of guests who order nothing. According to the I/E formula we have $N_0 = N - \sum N_i + \sum N_{ij} - \cdots$, where the numbers in the summations come from counting the people who ordered at least one thing (i.e. the numbers in the statement of the problem). But everyone orders something, so $N_0 = 0$, and we get $N = \sum N_i - \sum N_{ij} + \cdots$. Now we use the numbers above to compute $N = 140+190+100+90-(65+125+60+85+75+60)+(40+55+45+35)-10 = 215$.

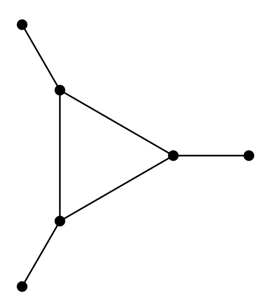

FIGURE 1.