1st exam: Oct 11 W 7–9pm.

Last time:

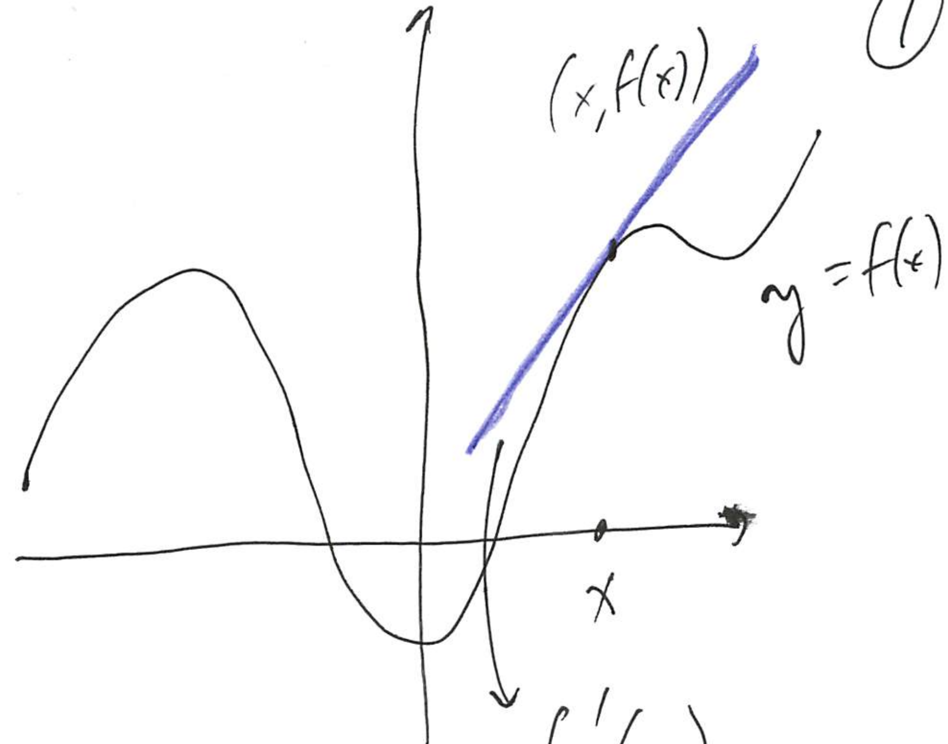
$f(x)$ function

$f'(x)$ derivative of $f(x)$

new function constructed from $f(x)$

its value at x is the slope of the tangent line to the graph of $f(x)$ at point $(x, f(x))$

(1)



$f'(x)$ computes the slope of the tangent line.

Definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>example</u> $f(x) = \frac{1}{x+1}$

compute $f'(x)$ using the definition of the derivative.

<u>ans</u> $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+1 - (x+h+1)}{(x+1)(x+h+1)}}{h}$$

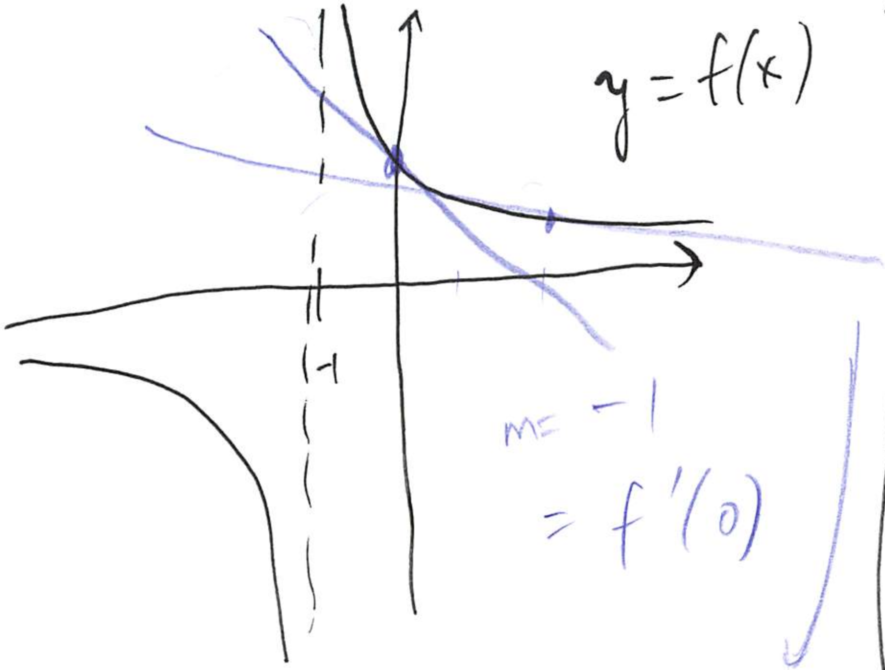
$$= \lim_{h \to 0} \frac{\cancel{x+1} - (\cancel{x}+\cancel{h}\cancel{+1})}{\cancel{h}(x+1)(x+h+1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)^2} = f'(x)$$

$$f(x) = \frac{1}{x+1}$$

$$f'(x) = -\frac{1}{(x+1)^2}$$



$m = -1$

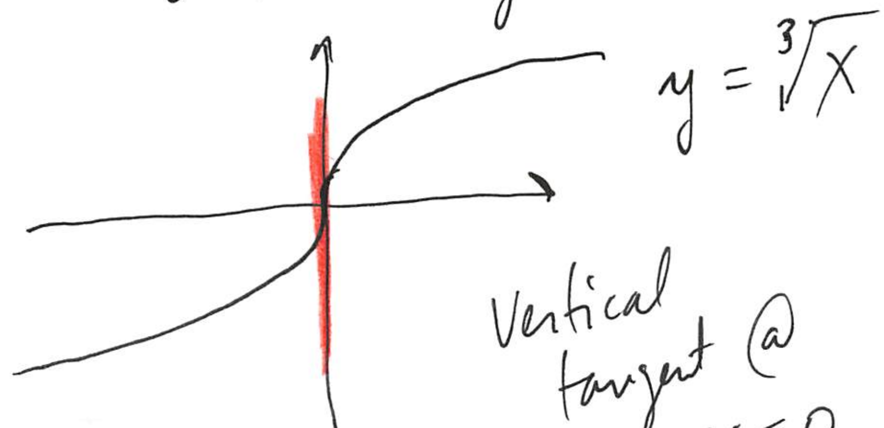
$= f'(0)$

$f'(2) = -\frac{1}{3}$

Domain of $f'(x)$ can be smaller than the domain of $f(x)$.

Various ways a graph can fail to have a tangent line at a point:

① graph could have a vertical tangent line

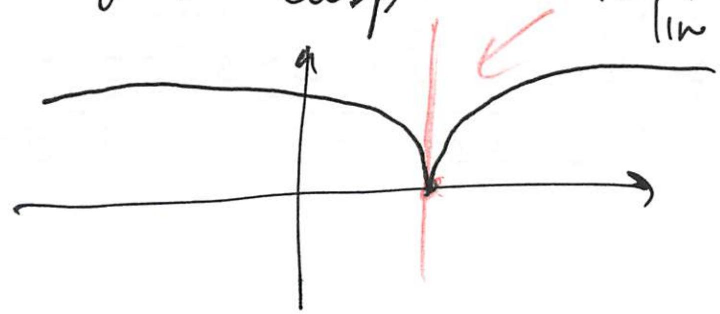


Vertical tangent @ $x = 0$

$\Rightarrow$ $x = 0$ not in domain of $f'(x)$

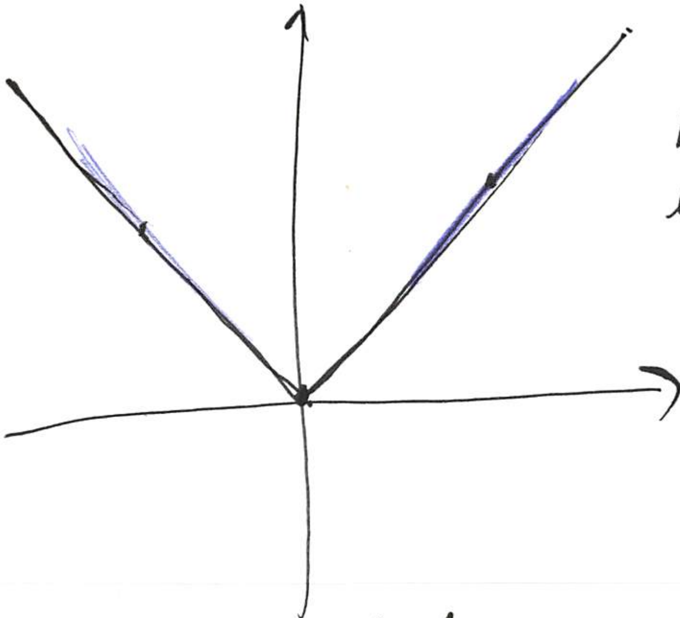
② $y = x^{2/3}$ graph could have a cusp.

no tangent line here.



③ graph can have a corner.

$f(x) = |x|$



no tangent line at $x=0$

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{not defined.} & x = 0 \end{cases}$$

④

We say that $f(x)$ is <u>differentiable</u> at $x=a$ if the derivative $f'(x)$ is defined at $x=a$

If $f(x)$ is differentiable at $x=a$, then automatically $f(x)$ is continuous at $x=a$. However, if $f(x)$ is continuous at $x=a$, this doesn't imply that $f(x)$ is differentiable at $x=a$.

consider $f(x) = |x|$ at $x=0$.

§ 3.1 Differentiation rules.

$f(x)$, derivative $f'(x)$ is defined using a limit.

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We want to be able to compute $f'(x)$ efficiently.

e.g. $f(x) = x^2$

$f'(x) = 2x$

e.g. $f(x) = \frac{1}{x+1}$

$f'(x) = -\frac{1}{(x+1)^2}$

<u>Goal</u> Build up a set of rules that allows us to compute the derivative of <u>anything</u>, without using the explicit definition.

<u>Notation</u>

$f(x)$ original function

$f'(x)$ derivative

$\frac{df}{dx}$ another version.

$\frac{d}{dx}(f(x))$

read it as

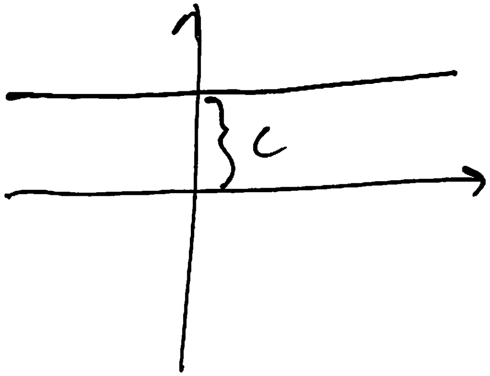
$\frac{d}{dx}$ means "derivative with respect to x"

Rules

1. constant function rule:

If $f(x) = C$, C = constant

then $f'(x) = 0$



← graph is horizontal line, so slope $= 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{C - C}{h}$$

$$= 0$$

2. power rule.

suppose $f(x) = x^n$, $n \geq 1$.

Then $f'(x) = nx^{n-1}$.

e.g. $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$

$\frac{d}{dx}(x^3) = 3x^2$

In fact power rule works for __any__ exponent, not necessarily an integer.

e.g. $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}})$

$$= \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}!$$

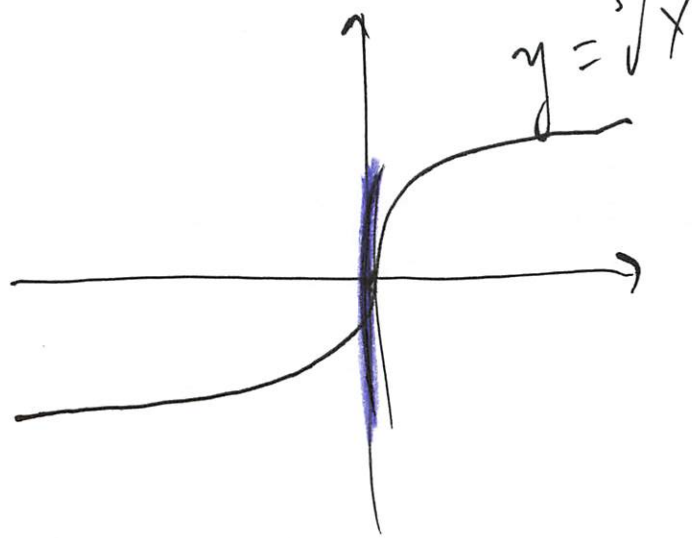
e.g. $\frac{d}{dx}\left(x^{\pi}\right) = \pi x^{\pi - 1}$

e.g. $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{1/3}\right)$

$$= \frac{1}{3} x^{-2/3}$$

$$= \boxed{\frac{1}{3\sqrt[3]{x^2}}}$$

$$\boxed{x^{p/q} = \sqrt[q]{x^p}}$$



3 sum/difference rule

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

e.g. $\frac{d}{dx}\left(x^2 + x^3\right)$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$$

$$= 2x + 3x^2$$

$$\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

e.g. $\frac{d}{dx}(x^5 - x^3)$

$$= 5x^4 - 3x^2$$

4 constant factor rule

C constant.

$$\frac{d}{dx}(C f(x))$$

$$= C \frac{df}{dx}$$

$$\frac{d}{dx}(7x^2)$$

$$= 7 \frac{d}{dx}(x^2)$$

$$= 7 \cdot 2x^1$$

$$= \boxed{14x}$$

using these rules, we can do many examples.

e.g. any polynomial.

$f = x^3 + 3x^2 - x + 4$

$f' = ?$

$$\frac{d}{dx}\left(x^3+3x^2-x+4\right)$$

sum/diff rule

$$=\frac{d}{dx}(x^3)+\frac{d}{dx}(3x^2)-\frac{d}{dx}(x)+\frac{d}{dx}(4)$$

2 4 2 1

$$=3x^2+3\frac{d}{dx}(x^2)-1+0$$

2

$$=3x^2+6x-1$$

e.g. $f(x)=x\sqrt{x}$

$$=x^1x^{1/2}$$

$$=x^{3/2}$$

$$f'(x)=\frac{3}{2}\cdot x^{1/2}$$

$$=\boxed{\frac{3}{2}\sqrt{x}}$$

e.g. $f(x)=x(x-1)$

$$=x^2-x$$

$$f'(x)=\boxed{2x-1}$$

Exponential functions

$$f(x)=a^x$$

a is a fixed number

$a>0$, called the "base" of the exponential.

e.g. $f(x)=2^x$

$$f(x)=3^x$$

$f(x) = 2^x$ is totally different from $g(x) = x^2$ <u>polynomial</u>.

$f'(x)$ is not computed using the power rule!

$f'(x) \overset{!}{\neq} x\,2^{x-1}$ X

Fact: $\frac{d}{dx}(a^x)$ is proportional to a^x.
The constant of proportionality depends on a.

$$\frac{d}{dx}(a^x) = C_a a^x$$

constant depending on a.

Why?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$= \underbrace{a^x}_{\text{orig fn}} \underbrace{\left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)}_{\text{some number that depends on } a}$$

e.g. $a = 2$

$$\lim_{h \to 0} \frac{2^h - 1}{h} = 0.693147\ldots$$

e.g. $a = 3$

$$\lim_{h \to 0} \frac{3^h - 1}{h} = 1.09861\ldots$$

Fact: there is a choice of base, in between 2 and 3, such that this limit equals 1.

This number is traditionally called $\boxed{e}$.

$$e = 2.718281828459045\ldots$$

not a rational number.

$$\Longrightarrow \frac{d}{dx}(e^x) = e^x \quad (*)$$

unique choice of base for exponentials to make (*) work.

Next time:

- product rule
- quotient rule
- derivatives of trig functions