MATH 611 EXAM

Let me know if you find any mistakes in the answers.

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution. You may quote results and examples used in class. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem. G always denotes a group. **Answer:** For a statement to be true, it must be true as stated with no additional words or hypotheses. If you find yourself saying things like, "Yes, that's true up to isomorphism," or "Yes that's true if you assume X," then it's not true, it's false!

- (a) (4 pts) Let G be a group. Then the map $\varphi\colon G \to G$ given by $\varphi(x) = x^{-1}$ is an automorphism.**Answer:** False. This is true only if G is abelian.
- (b) (4 pts) Any finite group has a unique composition series. **Answer:** False. Composition series typically aren't unique (consider Q_8, and other examples from HW). The successive quotients (composition factors) are unique up to a permutation but this is not what was asked.
- (c) (4 pts) Let N be a normal subgroup of a finite group G, and let $C \subset G$ be a conjugacy class. Then if $C \cap N \neq \emptyset$, we have $C \subset N$. **Answer:** True.
- (d) (4 pts) The alternating groups A_n, $n \geq 3$ are simple. **Answer:** False. A_4 is not simple.
- (e) (4 pts) Suppose G, H are abelian groups. Then any semi-direct product $G \rtimes H$ is abelian. **Answer:** False. The dihedral groups D_{2p} are semidirect products of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/p\mathbb{Z}$ for odd primes p. The nonabelian groups of order pq when $p|q-1$ are examples. There are many other examples.

(2) (20 pts) Let F be a field. Let $G \subset GL_3(F)$ be the matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a, b, c \in F.$$

- (a) (10 pts) Show that G is a group with respect to matrix multiplication. **Answer:** One just needs to check that the this subset is closed under inverses and taking products.

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(b) (10 pts) Suppose $F = \mathbb{Z}/2\mathbb{Z}$. Then G is a group of order 8, and from class we know there are five such groups up to isomorphism. Which one is it? **Answer:** We can first see that G is not abelian (this needs checking ... of course for general F it won't be, but something exceptional could happen when $F = \mathbb{Z}/2\mathbb{Z}$). This narrows it down to D_8 or Q_8. Then one can check that there are two elements of order 4, which means it must be D_8.

(3) (20 pts) Find all the normal subgroups of S_4. Be sure to justify your answer. **Answer:** Besides S_4 and the trivial subgroup, the other normal subgroups are A_4 and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We know A_4 is normal. Any normal subgroup must be union of conjugacy classes, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the cc of type 22 together with the identity, which has type 1111. All other unions of ccs can't form a subgroup. This is verified by checking their orders, or from seeing that once one cc is present others must be present too (e.g. if you have a 4-cycle you also have a product of two disjoint transpositions, etc.), and eventually you get S_4 or A_4.

(4) (20 pts) Let p be a prime.

(a) (6 pts) Define *finite p-group*. **Answer:** G is a finite p-group if the order of G is a p-power.

(b) (14 pts) Prove that if G is a finite p-group, then the center $Z(G)$ is nontrivial. **Answer:** Consider the class equation $|G| = |Z(G)| + \sum[G : C(g_i)]$, where the sum is taken over a finite set of representatives for the noncentral conjugacy classes. Take this equation mod p. The left vanishes, as does the sum (since they are noncentral, each index must be > 1, and so must be a nontrivial p-power). Thus $|Z(G)|$ must vanish mod p, which means it can't be 1.

(5) (20 pts)

(a) (8 pts) Prove that there are no simple groups of order 200. **Answer:** $200 = 2^3 \cdot 5^2$. We have $n_5 = 1 \pmod 5$ and $n_5 | 8$, so $n_5 = 1$ and the 5-Sylow is normal.

(b) (12 pts) Prove that there are no simple groups of order 56. **Answer:** $56 = 2^3 \cdot 7$. Thus $n_7 = 1$ or 8. If it's 1, then of course the 7-Sylow, which is isomorphic to $\mathbb{Z}/7\mathbb{Z}$, is normal. Otherwise there must be 48 elements of order 7, since any nonidentity element of $\mathbb{Z}/7\mathbb{Z}$ is a generator, and if there were any fewer some of the different 7-Sylows would coincide. But then there are only $56 - 48 - 1 = 7$ nonidentity elements left over, and together with the identity they must form the 2-Sylow, which has order 8. Thus the 2-Sylow is forced to be unique. Hence either $n_2 = 1$ or $n_7 = 1$, and the group can't be simple. Note: it turns out that that $n_7 = 1$ for *all* groups of order 48 (there are 52 of them). On the other hand $n_2 = 1$ about half the time.

(6) (20 pts)

(a) (5 pts) How many surjective group homomorphisms are there from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$? **Answer:** The kernel is a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and there are 3 of those.

(b) (5 pts) How many surjective group homomorphisms are there from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$? **Answer:** There are 6, since $|Aut(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})| = 6$ (we can freely permute the nonzero elements).

(c) (10 pts) How many (not necessarily injective) group homomorphisms are there from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to the dihedral group D_8 of order 8? **Answer:** The image must be trivial, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. D_8 has 5 elements of order 2 (refl. about the four diagonals and r^2) and 2 subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (two pairs of diagonals at right angles). If the image is $\mathbb{Z}/2\mathbb{Z}$ then there are 15 (3 choices of kernel and 5 choices of image.) If the image is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ there are 12 (pick the image and then pick one of the 6 ways to go to it.) Thus the total is $3\cdot 5+6\cdot 2+1=28$, where the last 1 accounts for the trivial hom.

(7) (20 pts) Up to isomorphism, how many finite abelian groups are there of order $n\le 32$? Justify your answer. **Answer:** For each $n\le 32$, we must compute the prime factorization $n=p_1^{e_1}\cdots p_k^{e_k}$ and then the number up to isomorphism for n is $N_n=P(e_1)\cdots P(e_k)$, where $P(e)$ is the number of partitions of e. For $e=1,2,3,4,5$, the partition numbers are $P(e)=1,1,3,5,7$. The set of pairs (n,N_n) looks like $(1,1)$, $(2,1)$, $(3,1)$, $(4,2)$, $(5,1)$, $(6,1)$, $(7,1)$, $(8,3)$, $(9,2)$, $(10,1)$, $(11,1)$, $(12,2)$, $(13,1)$, $(14,1)$, $(15,1)$, $(16,5)$, $(17,1)$, $(18,2)$, $(19,1)$, $(20,2)$, $(21,1)$, $(22,1)$, $(23,1)$, $(24,3)$, $(25,2)$, $(26,1)$, $(27,3)$, $(28,2)$, $(29,1)$, $(30,1)$, $(31,1)$, $(32,7)$ (one way to work through this list is to first do the squarefree numbers, then a prime squared times a squarefree number, etc.). The total number is 55.

(8) (20 pts) Let $G=GL_3(\mathbb{R})$, and let $H\subset G$ be the subgroup of diagonal matrices.

(a) (10 pts) Compute the normalizer $N_G(H)$. **Answer:** It consists of all matrices such that each row and column contains exactly one nonzero entry. This problem is challenging but doable, given what we know from class. There are several approaches.

First, it can be seen directly by computing the conjugate of a diagonal matrix by a generic 3×3 matrix. (Note that it is not neccessary to divide by the determinant ... in other words you don't have to use the inverse matrix in the conjugation, it suffices to use the adjugate matrix.) All off-diagonal elements must vanish, independent of what the original diagonal matrix is. By considering the possibilities one sees that this means that if an element in a given position is nonzero, everything else in its row and column must be zero. (What happens is each off-diagonal entry contains three products of triples of elements from the generic matrix. Looking at what triples occurs leads to the conclusion.)

A second method is to first look at $GL_2(\mathbb{R})$. It's easy to see that the normalizer there is the subgroup of off-diagonal matrices by direct computation. Then this subgroup appears in the normalizer of H in two different ways (upper-left corner, bottom-right corner). Taking products we get the matrices in $N_G(H)$ as above. Then if there is any other nonzero element, one can see from specific examples that it can't be in the normalizer.

Finally, here is a geometric way to see the answer, at least with $\mathbb{R}$ as coefficients. Let the diagonal matrix h have diagonal (s_1,s_2,s_3) and let A be the generic 3×3 matrix. Then we are looking for conditions to guarantee that AhA^{-1} is diagonal. The columns of A^{-1} form a dual orthonormal basis to the rows of A under the dot product (since $AA^{-1}=I$). We can think of this as the columns of A^{-1} determining three different planes that are perpendicular to the rows of A (for row i, take the plane spanned by columns j,k, where $\{i,j,k\}=\{1,2,3\}$). Now the condition that $(Ah)A^{-1}$ is diagonal for all h can be interpreted as the condition that the rows of Ah need to remain orthogonal to these planes for all h. If (x, y, z) is a row of A, then the corresponding row of Ah is $(s_1 x, s_2 y, s_3 z)$. Since the s_i are nonzero and arbitrary, it is clear that the only way this can work is if all but one of x, y, z is zero; otherwise the row can't stay orthogonal.

(b) (10 pts) Compute the quotient group $N_G(H)/H$. **Answer:** S_3. The cosets are represented by the permutation matrices.