CALCULUS 233H EXAM II

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

(a) (4 points) The gradient of $f(x,y) = 2x^2 + 3y$ is $4x+3$. **ANS:** False. This isn't even a vector.

(b) (4 points) $\lim_{(x,y)\to(0,0)} x^2/(x^2+y^2) = 1$. **ANS:** False. Set $y = kx$ and let x go to zero. The limit is $1/(k^2+1)$. Since this depends on k the limit doesn't exist.

(c) (4 points) If a particle moves with position function $\mathbf{f}(t)$, then its speed is given by $|\mathbf{f}'(t)|$. **ANS:** True.

(d) (4 points) A continuous function $f(x,y)$ attains a maximum and minimum value on a bounded region in the xy-plane. **ANS:** False. We need *closed and bounded*, not just *bounded*. For instance the function $f = x^2 + y^2$ has no maximum or minimum value on the open square $0 < x < 1, 0 < y < 1$.

(e) (4 points) The tangent plane to the graph of $xyz = 6$ at $(1,2,3)$ is given by $6x + 3y + 2z = 18$. **ANS:** True.

(2) (20 points) Find the curvature and torsion of $\mathbf{f}(t) = \langle t, t^2, t^2 + t^3\rangle$ at $t = 1$. **ANS:** Use the formulas $\kappa = |\mathbf{f}' \times \mathbf{f}''|/|\mathbf{f}'|^3$ and $\tau = (\mathbf{f}' \times \mathbf{f}'')\cdot \mathbf{f}'''/|\mathbf{f}' \times \mathbf{f}''|^2$. First compute the relevant derivatives, then plug in $t = 1$, then do the remaining computations. (In other words, don't take the lengths or products before plugging in $t = 1$.) We get $\kappa = \sqrt{13}/(15\sqrt{15})$ and $\tau = 3/26$.

(3) (20 points) Find the velocity, acceleration, and tangential and normal components of acceleration of the motion given by $\mathbf{f}(t) = \langle \cos t, \sin t, \cos(2t)\rangle$ at $t = \pi/4$. **ANS:** $\mathbf{f}' = \langle -\sin t, \cos t, -2\sin 2t\rangle$ is the velocity and $\mathbf{f}'' = \langle -\cos t, -\sin t, -4\cos 2t\rangle$. Plug in $t = \pi/4$ we get $\mathbf{f}' = \langle -1/\sqrt{2}, 1/\sqrt{2}, -2\rangle$ and $\mathbf{f}'' = \langle -1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$. The tangential component a_T is the component

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of $\mathbf{f}''$ in the direction of $\mathbf{f}'$. Since $\mathbf{f}' \cdot \mathbf{f}'' = 0$, we get $a_T = 0$. This means $\mathbf{f}''$ is all in the normal direction, so the normal component $a_N = |\mathbf{f}''| = 1$. If you prefer, you could just use the formulas in the textbook for a_T and a_N.

(4) (20 points) Find the maximum and minimum value of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$. **ANS:** This has two parts. First find the critical points of f on the interior $x^2 + y^2 < 1$ by solving $\nabla f = \mathbf{0}$. This gives $(1/2, 0)$. Next use Lagrange multipliers to find any candidates for max/min on the boundary $x^2+y^2 = 1$. So put $g(x, y) = x^2+y^2-1$ and solve $\nabla f = \lambda \nabla g$. We get 4 points $(1, 0)$, $(-1, 0)$, $(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$. (One equation gives $2y = \lambda y$, which means either $\lambda = 2$ or $y = 0$. You have to consider both possibilities, and this leads to the four points.) Plug all these points into f to find the max value is $9/4$ and the min value is $-1/4$. Note the min occurs in the interior, so if you forgot that step you didn't find the true min. (The graph of f is a paraboloid and that is its bottom.)

(5) (20 points) Find and classify the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$. **ANS:** We have to find the solutions of $\nabla f = \mathbf{0}$ and then classify them using the Hessian $D = f_{xx}f_{yy} - (f_{xy})^2$. (Finding the critical points is made easier by noticing that we must have $x = y$ for all of them (by the symmetry of f), but you don't need to notice that to get them.) The critical points are $(0, 0), (1, 1), (-1, -1)$. The first is a saddle and the next two are local mins.

(6) (20 points) Suppose that $z = f(x, y)$ and $x = s + t$, $y = s - t$. Prove that $(z_x)^2 - (z_y)^2 = z_s z_t$. **ANS:** We have $z_s = z_x x_s + z_y y_s$ and $z_t = z_x x_t + z_y y_t$ by the chain rule. From the equations we have $x_s = x_t = y_s = 1$ and $y_t = -1$. Thus $z_s z_t = (z_x + z_y)(z_x - z_y) = (z_x)^2 - (z_y)^2$.

(7) (20 points) Find the volume of the solid over the square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and under the graph of $z = 2 - x^2 - y^2$. **ANS:** This is asking for the double integral

$$\int_0^1 \int_0^1 (2 - x^2 - y^2)\, dx dy,$$

since the height function is $z = 2-x^2-y^2$ and the square is given by $0 \leq x \leq 1$, $0 \leq y \leq 1$. Doing the iterated integral gives $4/3$.