CALCULUS 233H EXAM I

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
  (a) (4 points) If $\mathbf{f}(t)$ is a vector-valued function representing position, then $\mathbf{f}'(t)$ is the velocity and $\mathbf{f}''(t)$ is the acceleration. **ANS:** True.
  (b) (4 points) The graph of $z^2 = x^2 + y^2$ is a sphere. **ANS:** False. This is a cone.
  (c) (4 points) If $\mathbf{a}$ and $\mathbf{b}$ are vectors in 3D and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b}$ is a nonzero vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. **ANS:** True. Note that $\mathbf{a}$ and $\mathbf{b}$ must themselves be nonzero for the assumptions to hold.
  (d) (4 points) The quantity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the volume of the parallelepiped with edges parallel to the 3D vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. **ANS:** False. This quantity can be negative. The volume is actually $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.
  (e) (4 points) If $\mathbf{a}, \mathbf{b}$ are vectors in three dimensions, then $|\mathbf{a} \times \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. **ANS:** True, since $|\sin\theta| \leq 1$ for any angle θ.

(2) (20 points) Let $\mathbf{f}(t) = \langle 6\ln t, 3t^2 - 6t, 2(2t+1)^{3/2}\rangle$. Compute the arc length of the graph of $\mathbf{f}$ from $t = 1$ to $t = e$. **ANS:** $3e^2+3$. Taking the length of $\mathbf{f}'$ we have $|\mathbf{f}'| = 6\sqrt{1/t^2 + 2 + t^2} = \sqrt{(1/t+t)^2}$. This equals $1/t+t$ on the interval $[1, e]$. The integral can then be done exactly.

(3) Let T be the triangle with vertices $(2, 0, 1)$, $(1, 0, 2)$, $(2, -1, 2)$.
  (a) (10 points) Verify that T is an equilateral triangle. **ANS:** By the distance formula, all sides have length $\sqrt{2}$.
  (b) (10 points) Find the area of T. **ANS:** Vectors along two sides are $\mathbf{a} = \langle 1, 0, -1\rangle$ and $\mathbf{b} = \langle 1, -1, 0\rangle$. The area is $|\mathbf{a} \times \mathbf{b}|/2 = \sqrt{3}/2$.

(4) (20 points) Find the distance from the plane $2x + 10y + 11z = 3$ to the line with parametric equations $x = -t + 1, y = -2t, z = 2t$. **ANS:** The distance is $|\hat{\mathbf{a}} \cdot \mathbf{b}|$ where $\mathbf{a} = \langle 2, 10, 11\rangle$ and $\mathbf{b} = \langle 1, -1, 1\rangle - \langle 1, 0, 0\rangle$. This gives $1/15$.

(5) (20 points) Let P_1 be the plane $x + 2y - z = 0$ and P_2 the plane $2x + 3y - 4z = -2$. Do P_1 and P_2 intersect? If they do, determine all the points in common. If they don't, explain why they don't. **ANS:** They intersect in a line since their normal vectors are not scalar multiples of each other. The direction vector is the cross product of the normal vectors and is $\langle -5, 2, -1\rangle$. A point on the line is $(1, 0, 1)$. We get $x = -5t + 1, y = 2t, z = -t + 1$ for the line of intersection.

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(6) (20 points) Verify that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ for all vectors $\mathbf{a}$, $\mathbf{b}$ in three dimensions. **ANS:** Long way: choose components for $\mathbf{a}$ and $\mathbf{b}$ and grind through it. Short way: divide both sides by $|\mathbf{a}|^2|\mathbf{b}|^2$. The LHS becomes $\sin^2\theta$ and the RHS becomes $1 - \cos^2(\theta)$.

(7) (20 points) Let H be the hyperboloid of one sheet with equation $x^2+y^2-z^2 = 1$. Suppose α and β are two real numbers that satisfy $\alpha^2 + \beta^2 = 1$. Show that the line L with parametric equations $x = \beta t + \alpha$, $y = -\alpha t + \beta$, $z = t$ is entirely contained in H. **ANS:** Plug in the equations of the line into the coordinates in the equation for H. Everything cancels and we get $0 = 0$, independent of t. Thus the line lies on H. There is actually another infinite family of lines on H too.

(8) (a) (10 points) Determine parametric equations for the curve C of intersection of the cylinder $y = x^2$ with the plane $x + y + z = 0$. **ANS:** $\mathbf{f}(t) = \langle t, t^2, -t - t^2\rangle$.

(b) (10 points) Write an integral that computes the arc length of C between the points $(0, 0, 0)$ and $(1, 1, -2)$, but do not evaluate the integral. **ANS:** $\mathbf{f}'(t) = \sqrt{1 + 4t^2 + (2t + 1)^2} = \sqrt{2 + 4t + 8t^2}$. The integral is then $\int_0^1 \sqrt{2 + 4t + 8t^2}\, dt$.

(c) (10 points extra credit – not part of the 100 points) Evaluate the integral. (*Hint:* $\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$.) **ANS:** You have to complete the square, then do a trig substitution, then use the formula I gave (or do integration by parts a few times). I leave you the pleasure of getting a final answer.