Problem 1. Show that the vector fields
\[ X_1 = z \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (x + y) \frac{\partial}{\partial z} \]
\[ X_2 = -z \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (-x + y) \frac{\partial}{\partial z} \]
define an involutive distribution \( \Delta \) in the positive orthant of \( \mathbb{R}^3 \). Find the 2-dimensional integral submanifolds of \( \Delta \).

Problem 2. Let \( X = (a_1(x, y, z), a_2(x, y, z), a_3(x, y, z)) \in \mathfrak{X}(\mathbb{R}^3) \) be a nowhere zero vector field. Consider the two-dimensional distribution \( \Delta(p) := (X(p))^\perp \)

1. Prove that \( \Delta \) is involutive if and only if:
\[
a_1 \left( \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - a_2 \left( \frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z} \right) + a_3 \left( \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) = 0
\]

2. Suppose there exists a function \( U(x, y, z) \) such that \( X = \vec{\nabla}U \). Show that \( \Delta \) is involutive and find the integral submanifolds of \( \Delta \).

Problem 3. Let \( X_1, X_2 \) be the vector fields in \( \mathbb{R}^3 \):
\[ X_1 = (1, 0, y) \quad X_2 = (0, 1, -x) \]

1. Prove that the distribution \( \Delta \) spanned by \( X_1 \) and \( X_2 \) is not involutive.

2. Let \( P \) and \( Q \) be arbitrary points in \( \mathbb{R}^3 \). Show that there exists a curve \( \alpha: \mathbb{R} \to \mathbb{R}^3 \) such that \( \alpha(0) = P, \alpha(1) = Q \) and for every \( t \in \mathbb{R} \):
\[ \alpha'(t) \in \Delta(\alpha(t)) \]