PROBLEM SET II SOLUTION HINTS

These are not complete solutions, just hints. If you have any questions, ask me in class or in person.

§7.4 (7) The quotient $R[x]/(x)$ is isomorphic to $R$, so both statements follow immediately from Props. 12 and 13.

(8) Since $a = ub$, we have $(a) \subseteq (b)$. Similarly $b = u^{-1}a$, so $(b) \subseteq (a)$ and thus $(a) = (b)$. Conversely, if $(a) = (b)$ then $b = xa$ and $a = yb$, which means $b(xy - 1) = 0$. Thus $x, y \in R^\times$ since $R$ is a domain.

(11) Suppose that $i \in I$ satisfies $i \notin P$. Then we must show that $J \subseteq P$. So select $j \in J$ and consider the product $ij$. We have $ij \in P$, since by hypothesis $I \cap J \subseteq P$, and $ij \in I \cap J$. Since $i \notin P$, we must have $j \in P$. Now apply this to all $j \in J$ to conclude $J \subseteq P$.

(13) (a) Suppose $Q = \varphi^{-1}(P)$ is not all of $R$. We know $Q$ is an ideal. Suppose $ab \in Q$ but $a, b \notin Q$. Then $\varphi(a)\varphi(b) \in P$ which means either $\varphi(a)$ or $\varphi(b) \in P$, a contradiction. If $\varphi$ is an inclusion homomorphism (this just means $\varphi$ is injective, so that we can identify $\varphi(R)$ with $R$), then $\varphi^{-1}(P)$ is $R \cap P$, which means the latter is a prime ideal.

(b) Suppose $Q = \varphi^{-1}(M)$ isn’t maximal, and let $I \supset Q$ be a proper maximal ideal containing it. The ideal $\varphi(I)$ must be all of $S$, since it contains $M$ and $M$ is maximal. But then $I = \varphi^{-1}(\varphi(I)) = \varphi^{-1}(S) = R$, which contradicts the maximality of $I$. For a counterexample, let $\varphi: \mathbb{Z} \to \mathbb{Z}$ be defined by $\varphi(x) = 0$. Then (2) is a maximal ideal in the target, but $\varphi^{-1}((2)) = (0)$, which isn’t maximal.

(16) (a) Use the division algorithm to write the given polynomial as $q(x^4 - 16) + r$, where the degree of $r$ is $\leq 3$.

(b) This follows from $x^4 - 16 = (x + 2)(x - 2)(x^2 + 4)$.

(26) Let $P$ be a prime ideal. If $r$ is nilpotent then $r^k = 0 \in P$, so $r \cdot r^{k-1} \in P$. If $r \notin P$, then $r^{k-1} \in P$. Hence $r \cdot r^{k-2} \in P$. Continuing by descending induction on $k$ we conclude $r \in P$. Since this is independent of the choice of $P$, we have $r \in \bigcap P$, where the intersection is taken over all prime ideals. Thus the nilradical is contained in the intersection of all prime ideals.

(37) If $R$ is local and $a \in R - M$, then we must have $(a) = R$. (The reason is that $(a)$ is either equal to $R$ or is contained in a maximal ideal, but the only maximal ideal is $M$.) Thus $1 \in (a)$ and $a$ is a unit. Conversely,
suppose the ideal $M$ of nonunits isn’t maximal, and let $N$ be the proper maximal ideal containing $M$. Then if $a \in N - M$ we have $(a) \subseteq N$. But $a$ must be a unit (since $M$ contains all nonunits), and thus $(a) = R$. Hence $M$ must be maximal. To see that $M$ is the unique maximal ideal, let $N$ be another maximal ideal and let $a \in N - M$. As before, $a$ must be a unit, and thus $(a) = R$. Since $(a) \subseteq N$, this contradicts the existence of $N$.

(38) This follows by applying Ex. 37.

§7.5  
(3) Consider the map $\mathbb{Z} \to F$ given by $1 \mapsto 1$. If this is injective, then $F$ contains $\mathbb{Q}$. If this has a kernel, then the image is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. But since $F$ is a field, we must have $n$ prime.

(5) By the previous HW set we know that the inverse of any nonzero formal power series is a Laurent series, which implies that the field of fractions of $F[[x]]$ is a subfield of $F((x))$ (the element $(f, g)$ of the field of fractions is mapped to the Laurent series $fg^{-1}$). We must show that every element of the Laurent series can be written as a ratio of two power series. To see this, given $f \in F((x))$, we take $g \in F[[x]]$ to be $1$ if the degree of $f$ is nonnegative, and to be $x^n$ if the degree of $f$ is $-n$. Then $fg \in F[[x]]$, so that $f$ corresponds to the element $(fg, g)$ of the field of fractions.

For the second part, suppose that there are two integral power series $f, g$ such that $f/g = h := \sum_{n \geq 0} x^n/n!$. Thus $gh$ is an integral power series. Write $g = \sum_{n \geq 0} a_n x^n$, and put $\sum_{i+j=n} a_i/j! = b_n$. Multiplying both sides of this equation by $n!$ we get $a_0 + na_1 + n(n-1)a_2 + \cdots + n!a_n = n!b_n$. If we reduce this modulo $n$ we get $a_0 = 0 \mod n$. Since this is true for all $n$, we must have $a_0 = 0$. Now repeat this argument using $a_1$ and $(n-1)!$ to see $a_1 = 0$. If we continue we conclude $a_n = 0$ for all $n$, and thus the exponential power series isn’t in the field of fractions of $F[[x]]$.

§7.6  
(4) If $a \neq 0$, then the element $(a, 0)$ is a zerodivisor since $(a, 0) \cdot (0, a) = (0, 0)$.

(5) (a) This uses that the map $\mathbb{Z} \to \prod \mathbb{Z}/n_i\mathbb{Z}$ is onto and that the $n_i$ are comaximal.

(b) Show that this element satisfies the congruences above by reducing it modulo $n_i$ for each $n_i$.

(c) 4377 and 15437

(6) For each power $x^k$, we can solve the congruences to determine the coefficient of $x^k$ in $f$. The second statement follows since in $\mathbb{Z} \to \prod \mathbb{Z}/n_i\mathbb{Z}$ we have $1 \mapsto (1, \ldots, 1)$.

(7) Use that fact that if $m = \prod p_i^{e_i}$, then $n = \prod p_i^{f_i}$ with each $f_i \leq e_i$. Then $\mathbb{Z}/m\mathbb{Z} \cong \prod \mathbb{Z}/p_i^{e_i}\mathbb{Z}$ and similarly for $\mathbb{Z}/n\mathbb{Z}$. Show that the map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is given on factors by $\mathbb{Z}/p_i^{e_i}\mathbb{Z} \to \mathbb{Z}/p_i^{f_i}\mathbb{Z}$. Now if $u \in \mathbb{Z}/p_i^{f_i}\mathbb{Z}$ is a unit, then $u$ is coprime with $p_i^{f_i}$. Now argue that any element in the
inverse image of $u$ must be coprime with $p_i^{e_i}$. Since the map is surjective on the units in each factor, it is surjective on the units globally.