This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then any four of the remaining problems.

You may cite results proved in class, in homework, or in the text in your solutions to the problems below, and need not reprove them. Otherwise (with the exception of Problem 1) you must provide complete proofs.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as True or False. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem. In this problem \( R \) denotes a commutative ring (not necessarily unital).

(a) An ideal \( I \) in \( R \) is a subring such that \( ri \in I \) for all \( i \in I \) and all \( r \in R \). \textbf{ANS:} True.

(b) If \( R \) is unital, then the unit subgroup \( R^\times \subset R \) forms a subring of \( R \). \textbf{ANS:} False. \( 0 \not\in R^\times \).

(c) If \( R \) is a unique factorization domain, then \( R \) is a principal ideal domain. \textbf{ANS:} False.

(d) If \( S \) is a noncommutative ring, then there can exist no homomorphisms \( \varphi: R \to S \) and \( \phi: S \to R \). \textbf{ANS:} False. We can take these to be the zero maps.

(e) If \( R \) is a principal ideal domain, then \( R \) is Noetherian. \textbf{ANS:} True.

(f) An element \( r \in R \) is a zerodivisor if and only if \( r \) is nilpotent. \textbf{ANS:} False. \( 2 \in \mathbb{Z}/(6) \) is a zerodivisor that isn’t nilpotent.

(2) Let \( R \) be a commutative ring, not necessarily unital, and let \( I, J \) be ideals in \( R \). For each statement, either prove it or show by a counterexample that it is false.

(a) The intersection \( I \cap J \) is an ideal of \( R \). \textbf{ANS:} This statement is true and is easy to verify.

(b) The union \( I \cup J \) is an ideal of \( R \). \textbf{ANS:} False. Take \( R = \mathbb{Z} \), \( I = (2) \), \( J = (3) \). Then \( 2 + 3 \not\in I \cup J \).

(3) (a) Give an example of a ring with a prime ideal that isn’t maximal. \textbf{ANS:} Let \( R = \mathbb{Z}[x] \) and \( I = (x) \). Then \( \mathbb{Z}[x]/(x) \cong \mathbb{Z} \), which is an integral domain but not a field. Another example from HW is given by \( R = \mathbb{Q}[x, y] \) and \( I = (x) \).

(b) Give an example of a unique factorization domain that isn’t a principal ideal domain. \textbf{ANS:} We proved in class that \( \mathbb{Z}[x] \) is a unique factorization domain. But \( \mathbb{Z}[x] \) isn’t a principal ideal domain, since we showed in class that \( (2, x) \) isn’t principal.

(4) Let \( \varphi: R \to S \) be a homomorphism, and define a map \( \bar{\varphi}: R[[x]] \to S[[x]] \) by \( \bar{\varphi}(\sum_{k \geq 0} a_k x^k) = \sum_{k \geq 0} \varphi(a_k)x^k \). Show that \( \bar{\varphi} \) is a homomorphism. \textbf{ANS:} We must show \( \bar{\varphi}(f + g) = \bar{\varphi}(f) + \bar{\varphi}(g) \) and \( \bar{\varphi}(fg) = \bar{\varphi}(f)\bar{\varphi}(g) \). The first is easy to check, since \( \bar{\varphi} \) is defined by applying \( \varphi \) to each term, and addition is done term by term. For the second, let \( f = \sum a_i x^i \), \( g = \sum b_j x^j \). Then \( fg = \sum_k (\sum_{i+j=k} a_i b_j) x^k \), and \( \bar{\varphi}(fg) = \sum_k \varphi(\sum_{i+j=k} a_i b_j)x^k = \sum_k (\sum_{i+j=k} \varphi(a_i)\varphi(b_j))x^k = \bar{\varphi}(f)\bar{\varphi}(g) \).

(5) Let \( R \) be a commutative unital ring. Show that \( f = \sum_k a_k x^k \in R[x] \) is nilpotent if the \( a_k \) are nilpotent. \textbf{ANS:} It is clear that \( ax^k \) is nilpotent if \( a \) is nilpotent. We proved in HW that the sum of two nilpotent elements is nilpotent. (In fact the set of nilpotent elements forms an ideal.) Thus \( \sum_k a_k x^k \) is nilpotent if the \( a_k \) are. Another solution is to prove that if \( N \) is large enough, then every

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monomial in \( f^N \) contains an \( a_k \) to a sufficiently large power, so that \( f^N = 0 \). (This could be done by induction on the degree of \( f \).) The converse of this result is also true, and is proved as follows. If \( f \) is nilpotent, then the coefficient of its leading term must be nilpotent. Since the nilpotent elements form an ideal, we can subtract the leading term to get a polynomial of lower degree that must also be nilpotent. Now we use induction to complete the proof.

(6) Let \( R \) be a commutative unital ring, and let \( I, J \) be two ideals in \( R \). The \textit{ideal quotient} \( (I : J) \) is defined to be \( \{ x \in R \mid xj \in I \text{ for all } j \in J \} \). Show that \( (I : J) \) is an ideal. \textbf{ANS:} Obviously \( 0 \in (I : J) \), since \( 0 \) is in any ideal. If \( x, y \in (I : J) \), then \( (x + y)j = xj + yj \in I \) for all \( j \), and similarly for \( -x \). Finally if \( x \in (I : J) \) and \( r \in R \), then \( rxj = r(xj) = rj \) for some \( i \in I \). Thus \( rx \in (I : J) \), since \( I \) is an ideal.

(7) Let \( H \) be the rational quaternions, that is \( H = \{ q_1 + q_2i + q_3j + q_4k \mid q_i \in \mathbb{Q} \} \), where \( i^2 = j^2 = k^2 = -1 \), and \( ij = -ji = k, jk = -kj = i, \) \( ki = -ik = j \). Let \( e_1 = 1, e_2 = i, e_3 = j, \) and \( e_4 = (1 + i + j + k)/2 \). Show that the subset \( R \subset H \) of all elements of the form \( \sum n_i e_i \), where \( n_i \in \mathbb{Z} \), is a subring. \textbf{ANS:} \( R \) contains 0 and is easily seen to be an abelian subgroup. We must show that \( R \) is closed under multiplication. By linearity, it suffices to prove \( e_i e_j \in R \) for all \( i, j \). Note that \( k = 2e_4 - e_1 - e_2 - e_3 \in R \). Now since we know that \( H \) is a ring, we only have to consider the products \( e_i e_4 \) and \( e_4 e_i \) for \( i = 2, 3, 4 \). These are easy to compute explicitly. The hardest is \( e_2 e_4 \), which turns out to be \( e_4 - e_2 \).

(8) Let \( R \) be a commutative unital ring. The \textit{Jacobson radical} \( R_J \) of \( R \) is defined to be the intersection of all maximal ideals of \( R \). Show that if \( x \in R_J \), then \( 1 - xy \) is a unit for all \( y \in R \). \textbf{ANS:} Suppose \( 1 - xy \) isn’t a unit for some \( y \in R \). Then the ideal \( (1 - xy) \) is a proper ideal, and is thus contained in some maximal ideal \( M \). Now \( xy \in M \), since \( x \) is in the intersection of all maximal ideals. But \( M \) is an additive subgroup, so \( 1 - xy \in M \) and \( xy \in M \) implies \( 1 \in M \). Hence \( M = R \), a contradiction.