Advanced Calculus Midterm
due 3/24/17

Show all of your work and give full explanations. Write legibly and hand back your exam stapled.

Problem 1. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2\,, \qquad f(x_1, x_2) = (x_1^3 - 3x_1x_2^2, 3x_1^2x_2 - x_2^3)$$

(i) Determine the largest subset of points $U \subset \mathbb{R}^2$ on which $Df(x)$ is invertible.

(ii) Provide an argument why around each $\mathring{x} \neq 0$ there exists an open disk $B = B(\mathring{x}, r)$ of some radius $r > 0$ on which $f\colon B \to f(B)$ is a diffeomorphism. Calculate $D(f^{-1})(f(x))$ for $x \neq 0$ and evaluate your result at the point $\mathring{x} = (1, 1)$.

(iii) Show that for each $x \neq 0$ the curves (both of which go through the point $f(x_1, x_2)$)

$$t \mapsto f(x_1 + t, x_2) \quad \text{and} \quad t \mapsto f(x_1, x_2 + t)$$

obtained by mapping the x_1 and x_2 lines by f, are perpendicular (i.e. their tangent vectors are perpendicular) at $t = 0$.

(iv) Is it true that then also the images of the y_1 and y_2 lines under f^{-1} are perpendicular? Provide a proof of your answer.

Problem 2. Consider the "wormhole" in 4 dimensions given by the equation

$$x_2^2 + x_3^2 + x_4^2 = (\cosh x_1)^2$$

(i) Show that the above equation defines a 3-dimensional submanifold of $\mathbb{R}^4$.

(ii) What are the level surfaces $x_1 = c$ constant for various values of $c \in \mathbb{R}$?

(iii) Picture this 3-dimensional surface by drawing an accurate picture of the 2-dimensional surface you obtain by setting $x_2 = 0$. What becomes of the original x_1 level surfaces in this dimensional reduction?

(iv) Calculate a (non-zero) normal vector (does not need to have length 1) for the 3-dimensional tangent plane in $\mathbb{R}^4$ at any point x on the wormhole.

(v) Check that $\mathring{x} = (0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ lies on the wormhole and calculate the tangent plane at this point in the form $\mathring{x} + \operatorname{span}\{v_1, v_2, v_3\}$ where $v_i \in \mathbb{R}^4$ are three linearly independent vectors in $\mathbb{R}^4$.

Problem 3. The heat distribution of a star positioned at $x = 0$ in $\mathbb{R}^3$ is given by the usual Gaussian $T(x) = e^{-||x||^2}$. You are flying a space probe along a parabolic trajectory $\gamma(t) = (t, t^2 + a, 0)$ in the x_1, x_2 plane for various choices of $a \in \mathbb{R}$.

(i) Draw the typical trajectory for different values of $a \in \mathbb{R}$ in the x_1, x_2 plane.

(ii) Show that for $a \geq -1/2$ there is exactly one maximum for the temperature along the trajectory. Where does it occur and what is the temperature there?

(iii) Show that for $a < -1/2$ there are one local minimum and two maxima. Where do they occur and what is the temperature there. How small can the temperature get along the trajectory? Explain why the minimum is only a local minimum.

(iv) Verify that at all points $x \in \mathbb{R}^3$ on the trajectory where there is a (local) maximum/minimum the trajectory is tangent to the circle of radius $||x||$ centered at the origin. (Bonus: provide a conceptual - without any computations - reason why this has to be so).

Problem 4. Consider the equations

$$x_1x_3 - x_2x_4 = 1\,, \qquad x_2x_3 + x_1x_4 = 0$$

on $\mathbb{R}^4$.

(i) Show that these equations define a 2-dimensional submanifold (surface) $M \subset \mathbb{R}^4$.

(ii) Show that the point $\mathring{x} = (0, 1, 0, -1) \in M$ and calculate the tangent plane $\mathring{x} + T_{\mathring{x}}M$ by giving a basis for $T_{\mathring{x}}M$.

(iii) Slice this surface M by the 3-dimensional plane $x_4 = 0$. One expects to get a curve. If so, give a formula for this curve and draw it.

(iv) Find explicitly a map $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^4$ describing M as a graph:

$$M = \{(x_1, x_2, f(x_1, x_2))\,;\, (x_1, x_2) \neq (0, 0)\}$$