HOMEWORK 9, HONORS CALCULUS II
11/8/2018

Problem 1. Show that a differentiable function $f(x)$ which satisfies the condition $f'(x) = f(x)$ must be of the form $f(x) = ce^x$ for some constant $c \in \mathbb{R}$. *Hint*: show that $f(x)e^{-x}$ is constant, by showing that its derivative is zero for all values of x.

Problem 2. Here a purely algebraic way (in contrast to the calculus type proof given in class using the first problem above) to verify the functional equation of the exponential function, namely $e^{x+y} = e^x e^y$. For us

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Here some hints: you will have to use the binomial formula (which you probably have learned in high school, otherwise learn it now)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Then you will have to stare at the summands a bit, perhaps interchange sums, rearrange terms, rename summation indices, and combine binomial coefficients— this is pretty hard to do, so try this only when you feel under-challenged in the class...

Problem 3. Using the definition of $a^x = e^{x \ln a}$ for $a > 0$ and the properties of e^x and $\ln x$, calculate/verify the following formulas (which you probably have seen in high school calculus):

(i) $(a^x)' = ?$
(ii) $\ln a^x = x \ln a$
(iii) $(\log_a x)' = ?$ if $\log_a x$ denotes the inverse function of a^x, that is $a^{\log_a x} = x$ and $\log_a(a^x) = x$. Draw the graphs of the function a^x for $0 < a < 1$, $a = 1$, and $a > 1$.
(iv) $(x^x)' = ?$

Problem 4. One version of Zeno's Paradox is the following: suppose I wish to cross a room of a certain length. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance...and so on forever. The consequence is that I can never get to the other side of the room, that is, it would take infinitely long. Since everyday experience shows that we can actually cross a room in finite time, something must be wrong with Zeno's reasoning. Can you resolve the paradox?

Problem 5. You compound an initial investment of 1 with annual interest rate x (e.g. $x = 0.03$ would be a 3% rate) n-times through the year.

(i) Show that your initial investment of 1 grew to $(1 + \frac{x}{n})^n$ after one year.
(ii) Following Euler, use the binomial formula to expand the above expression and calculate the limit $n \to \infty$ of $(1 + \frac{x}{n})^n$. What do you get? interpret your result in terms of compounding.

Problem 6. Use the following ideas to derive a power series expansion for $\ln(1+x)$:

(i) Write $\ln(1 + x) = \int ?\, dx$.

(ii) Expand the integrant ? in a geometric series and then integrate term by term to obtain

$$\ln(1+x) = \sum_{k=0}^{\infty} a_k x^k$$

for some specific numbers a_k which you will find when carrying out those steps.

(iii) Show that your power series converges as long as $|x| < 1$. Calculate an approximation of $\ln(3/2)$ by summing the first 6 terms of the power series. How does this number compare to the value of $\ln(3/2)$ you get from your calculator?

Problem 7. Here an idea of Newton's who wrote: "All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded mathematics and philosophy more than at any other time since". For a real number $\alpha > 0$ consider the power series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where, in agreement with the coefficient of the binomial formula, we define

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

Show the following:

(i) If $\alpha = n$ is a positive integer, then the above series is a finite sum. Identify this sum as an expression you know well.

(ii) If $\alpha > 0$ is *not a natural number*, i.e., the series is not a finite sum, show that the power series converges for $0 \le x < 1$.

(iii) Extrapolating from (i), it was clear to Newton that

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

holds for any real number α. This *generalized binomial formula* is engraved on his tomb in London. Use this formula to approximate $\sqrt{3/2}$ by summing the first 4 terms of the series and compare this to the "actual" value of $\sqrt{3/2}$ given by the calculator.

Problem 8. Determine and provide a proof whether the following series of numbers converge or diverge:

(i) $\sum_{k=1}^{\infty} n^3 2^{-n}$

(ii) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

(iii) $\sum_{k=1}^{\infty} \frac{n^2+3n-5}{1+n^2}$

(iv) $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$

(v) Write $0.6666666\ldots$ periodic as an infinite series and show that this series converges to $2/3$.

(vi) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

(vii) $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(2n^2+5)}$

(viii) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$