Homework 8, Differential Geometry
due 4/21/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. In this problem we try to find the critical points (ideally minima) of the energy functional on curves in the hyperbolic plane. Generally, for any Riemannian manifold (U, g) the energy functional is given by

$$E(\gamma) = \tfrac{1}{2} \int_I g_{\gamma(t)}(\gamma'(t), \gamma'(t))dt$$

for curves $\gamma\colon I \to U$. For those who have some rudimentary physics background, the energy functional is just the total kinetic energy over the trajectory (of some particle of mass $= 1$). Thus we are dealing with "free particle motion" in the geometry determined by g. Physics tells us that free particles travel in such a way as to minimize the total energy along their trajectories. For instance, if we are in Euclidean space then the free particle travels along a straight line with uniform speed. As we have pointed out before, the energy functional is *not* parameterization invariant, thus the minimizers come with a god given parameterization (unlike in the case of the length functional, where the choice of parameterization does not matter). This means that in computations we are not allowed to make any assumptions on the parameterization (i.e., we cannot assume arclength parameterization if we are interested in the actual physics of the situation, and not just in the trace the particle makes, say, in a cloud chamber).

(i) For the upper half plane model of hyperbolic space (see HW 7) calculate the ODE a curve $\gamma\colon I \to H^2$ has to satisfy in order to be a critical point of the energy functional under compactly supported variations.

(ii) Verify that vertical lines in H^2 and circles meeting the x-axis at a right angle (how parameterized?) are solutions of your ODE.

(iii) Recall that $\mathbf{SL}(2, \mathbb{R})$ acts by isometries on H^2 via

$$A \cdot z = \frac{az + b}{cz + d}$$

for $A \in \mathbf{SL}(2, \mathbb{R})$. Show that the energy does not change when a curve γ is moved by an isometry to $A \cdot \gamma$.

(iv) Let $V = \dot\gamma$ where the variation $\gamma_s = A(s) \cdot \gamma$ is given by a curve $s \mapsto A(s)$ in $\mathbf{SL}(2, \mathbb{R})$ with $A(0) = I_2$. In other words, we choose as a variation the curve γ moved by a 1-parameter family of isometries. Show that whenever X is a trace free 2×2 matrix, then $A(s) = \exp(sX) \in \mathbf{SL}(2, \mathbb{R})$ and calculate V for the cases

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If you wonder why exactly those 3 matrices X, verify that they form a basis of the vector space of trace free 2×2 matrices.

(v) Now assume γ is a critical point for the energy functional and V comes from a 1-parameter group of isometries as above. Show that

$$< V, \gamma' >= c < \gamma, e_2 >^2$$

for some constant $c \in \mathbb{R}$ where $e_2 = (0,1)$ is the second standard basis vector.

(vi) Use the special V coming from the 3 matrices X above and try to calculate the critical curves γ. Ideally you should be able to deduce (ii).

Problem 2. Let $U \subset \mathbb{R}^n$ be an open subset (a local manifold). The *vector field Lie bracket* of two vector fields $X, Y\colon U \to \mathbb{R}^n$ so defined by

$$[X,Y](p) = X(p)\cdot Y - Y(p)\cdot X \qquad p \in U$$

where, for any smooth function $f\colon U \to \mathbb{R}$ and vector $v \in \mathbb{R}^n$ at $p \in U$ the notation $v \cdot f := df_p(v)$ denotes the directional derivative of f at p along v. If X is a vector field, then $X \cdot f\colon U \to \mathbb{R}$ is the function $(X\cdot f)(p) = X(p)\cdot f$. For a vector valued function these notations are applied to each component. Thus, in the usual notation the Lie bracket is

$$[X,Y](p) = dY_p(X(p)) - dX_p(Y(p))$$

for $p \in U$. So, starting with two vector fields X, Y the Lie bracket $[X,Y]$ gives a new vector field.

(i) Show that $[X,Y]$ is skew symmetric bilinear in X and Y over $\mathbb{R}$.

(ii) For smooth functions $f, g\colon U \to R$ show that

$$[fX, gY] = fg[X,Y] + f(X\cdot g)Y - g(Y \cdot f)X$$

Notice that the product of a function and a vector field is defined pointwise, i.e., $(fX)(p) = f(p)X(p)$ for $p \in U$.

(iii) Show that the Lie bracket of constant vector fields is zero.

(iv) Calculate a formula of the Lie bracket in terms of the components of the vector fields in a basis $\{e_i\}$ of $\mathbb{R}^n$, that is if $X = \sum_{i=1}^n \xi_i e_i$, $Y = \sum_{i=1}^n \eta_i e_i$ then find the components ζ_i of $[X,Y] = \sum_{i=1}^n \zeta_i e_i$.

(v) Show that $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$. This is called the *Jacobi identity*.

(vi) Now let $\alpha \in \Omega^1(U,\mathbb{R})$ be a 1-form on U and $X, Y\colon U \to \mathbb{R}^n$ vector fields. Verify that the exterior derivative can be calculated via

$$d\alpha(X,Y) = X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y])$$

where the smooth function $\alpha(X)\colon U \to \mathbb{R}$ is given by $\alpha(X)(p) := \alpha_p(X(p))$ for $p \in U$.

Problem 3. Let $U := \mathbb{R}^2 \setminus \{0\}$ be the punctured plane and γ the counter clockwise once traversed unit circle. Show that the map

$$[\alpha] \mapsto \int_0^{2\pi} \alpha_{\gamma(t)}(\gamma'(t))dt$$

between the first deRham cohomology group $H^1_{dR}(U,\mathbb{R})$ and $\mathbb{R}$ is a linear isomorphism. Here $[\alpha]$ with $d\alpha = 0$ denotes the coset in the abelian group

$$H^1_{dR}(U,\mathbb{R}) = \frac{\operatorname{kernel} d\colon \Omega^1 \to \Omega^2}{\operatorname{image} d\colon \Omega^0 \to \Omega^1}$$

Convince yourself that this map is well-defined, i.e., independent of the choice of representative of the coset $[\alpha]$. In other words, if $\alpha = df$ then $\int_0^{2\pi} \alpha_{\gamma(t)}(\gamma'(t))dt = 0$.