Homework 8, Advanced Calculus
due 4/14/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the quadratic function

$$f(x) = x^t A x + b^t x + c$$

on $\mathbb{R}^n$, where $A = A^t$ is a symmetric matrix, $b \in \mathbb{R}^n$ is a column vector, and $c \in \mathbb{R}$.

(i) Calculate the gradient of f (We did this in class, but you should do it yourself).

(ii) If $\det A \neq 0$ show that there exists $\alpha \in \mathbb{R}^n$ so that

$$f(x+\alpha) = x^t A x + \tilde{c}$$

What is α and $\tilde{c}$ in terms of A, b and c? (This justifies, in the case when A is invertible, to only study the case for which $b = 0$ in the quadratic function f).

Problem 2. The best fitting straight line–least square method. Given n data points $(x_i, y_i) \in \mathbb{R}^2$ in the plane, $i = 1, \ldots, n$, which one knows (from some theory) should roughly lay on a straight line $y = mx + b$ the problem is to find the values for the slope m and y-intercept b which makes the line a "best fit to the data (x_i, y_i)". By this one means that the sum of the squares of the vertical distances of the points (x_i, y_i) to the line are *minimized.*

(i) Find the conditions on m and b so that the line $y = mx + b$ is a best fit to the given data (x_i, y_i), $i = 1, \ldots, n$, and provide an argument why these values for m and b minimize the sum of the squares of the vertical distances of the points (x_i, y_i)

(ii) If $y = mx + b$ is the best fitting line, show that $\sum_{i=1}^n (mx_i + b - y_i) = 0$, which means that the positive and negative vertical distances cancel out (which may be expected for a best fitting line).

(iii) If the data set consists of two points (x_1, y_1) and (x_2, y_2) show that the best fitting line you calculated in (i) is just the line connecting those two points.

(iv) Find the best fitting line for the data set $(0,1), (1,3), (2,2), (3,4), (4,6)$. Draw the data points and the best fitting line and hopefully you see the statement of item (ii).

Problem 3. Let $M = \{x \in \mathbb{R}^n\,;\, f(x) = 0\}$ be a hypersurface described as the zero set of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with the property that $Df(x) \neq 0$ for all $x \in M$. Assume that there is a point $\mathring{x} \in M$ for which the distance to the origin of points on M is minimized. Show that the line connecting the origin and $\mathring{x}$ is perpendicular to M at $\mathring{x}$.

Problem 4. Consider the ellipse $(x/a)^2 + (y/b)^2 = 1$. What is the dimension of the inscribed rectangle of largest area?

Problem 5. This is the standard life guard dilemma: a person in the water is needing help and the life guard is trying to get there in the least amount of time. The assumption is that the life guard runs and swims along a straight line. Her speed running on sand is v_1 and her swimming speed v_2. Find the trajectory the life guard is taking to minimize the time to get to the person needing help in the water. In particular, find a relation among v_1, v_2 and the two angles α_1, α_2 the straight lines (when moving on the sand, and when swimming in the water) make with the shore line. Compare your formula to Snell's Law governing the propagation of light when it passes from one medium to another.