Homework 7, Differential Geometry
due 4/7/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the upper half plane $H^2 = \{z = x + iy \in \mathbb{R}^2\,;\, y > 0\}$ with Riemannian metric $g = \frac{1}{y^2} <,>$ where $<,>$ is the standard inner product on $\mathbb{R}^2$. For each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det A = 1$, i.e. $A \in \mathbf{SL}(2,\mathbb{R})$, consider the action

$$A \cdot z = \frac{az + b}{cz + d}$$

Show:

(i) The x-axis is mapped into the x-axis under the action of $A \in \mathbf{SL}(2,\mathbb{R})$ and the upper half plane H^2 is mapped into the upper half plane.
(ii) The action of $A \in \mathbf{SL}(2,\mathbb{R})$ is an isometry for g.

Problem 2. On $\mathbb{C}^2$ consider the indefinite inner product $<< v, w >>= v_1\bar{w}_1 - v_2\bar{w}_2$ for $v, w \in \mathbb{C}^2$. The special unitary group for this inner product is notated by $\mathbf{SU}(1,1)$. In other words, $A \in \mathbf{SU}(1,1)$ if and only if A is a 2×2 complex matrix with $<< Av, Aw >>=<< v, w >>$ and $\det A = 1$. Let $D = \{|z|^2 < 1\}$ denote the (open) unit disk with Riemannian metric $g = \frac{4}{(1-|z|^2)^2} <,>$ where $z = x + iy \in \mathbb{R}^2 = \mathbb{C}$ and $<,>$ is the standard dot product on $\mathbb{R}^2 = \mathbb{C}$. Define the action

$$A \cdot z := \frac{az + b}{cz + d}$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SU}(1,1)$ and $z \in \mathbb{C}$. Show:

(i) The unit circle $|z|^2 = 1$ is preserved by the action of $A \in \mathbf{SU}(1,1)$, the interior of the unit circle (the open unit disk D) is mapped into D and the exterior of the unit circle is mapped into itself.
(ii) The action of $A \in \mathbf{SU}(1,1)$ is an isometry for g.

Problem 3. The Riemannian manifolds H^2 and D with their respective metrics from the previous problems are isometric. Find an explicit isometry.
Hint: the map should "bend" the x-axis in the H^2 picture to the unit circle in the D picture etc.

Problem 4. Parametrize the unit sphere $S^2 \subset \mathbb{R}^3$ by latitude and longitude angles $f\colon U \to S^2$ and calculate the induced Riemannian metric.

Problem 5. (i) Calculate the stereographic projection of the unit sphere S^2 from the north pole into the equatorial plane $\mathbb{R}^2 = \{z = 0\}$, i.e. find a formula for the map

$$\varphi\colon S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$$

so that $\varphi(p)$ is the unique intersection point of the line through $(0,0,1)$ and $p \in S^2$ with the equatorial plane.
(ii) Calculate the inverse of φ and show that this gives a parametrization $f\colon \mathbb{R}^2 \to S^2$ missing the northpole.

(iii) Calculate the induced metric of the 2-sphere under the map f.