Homework 6, Advanced Calculus
due 3/8/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the map $f(x_1, x_2) = (x_1 x_2, x_1^2 - x_2^2)$ from class.

(i) Show that f as a map $f\colon U \to V$ where $U = \{x \in \mathbb{R}^2\,;\, x_1 > 0, x_2 > 0\}$ and $V = \{y \in \mathbb{R}^2\,;\, y_1 > 0\}$ is bijective.
(ii) Calculate the inverse function $f^{-1}\colon V \to U$, i.e. find an explicit formula $f^{-1}(y_1, y_2) = ?$
(iii) Verify that both f and f^{-1} are differentiable. Thus $f\colon U \to V$ is a diffeomorphism.
(iv) Draw the y_1 and y_2 lines under the map f^{-1}, i.e. the curves $y_1 \mapsto f^{-1}(y_1, y_2)$ for $y_2 = c$ constant and similar for the y_2 lines.

Problem 2. Consider the spherical coordinates

$$f(r, \alpha, \beta) = (r\cos\alpha\cos\beta, r\sin\alpha\cos\beta, r\sin\beta)$$

on $\mathbb{R}^3$.

(i) Calculate $Df(r, \alpha, \beta)$ and its determinant $\det Df(r, \alpha, \beta)$.
(ii) Calculate the image $V = f(U)$ under f of the open domain $U = (0, \infty) \times (0, 2\pi) \times (-\pi/2, \pi/2)$.
(iii) Show that $f\colon U \to V$ is a diffeomorphism, i.e., f is bijective and f, f^{-1} are differentiable.
(iv) Draw the r, α and β lines under f.

Problem 3. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear map $f(x) = Ax$ given by an $m \times n$ matrix A. Give an argument why f is differentiable and calculate $Df(x)$.

Problem 4. Let $A, B\colon \mathbb{R} \to M_n$ be two differentiable functions of $t \in \mathbb{R}$ with values in $n \times n$ matrices. Calculate a formula for the derivative of the matrix product $\frac{d}{dt}(AB)$ in terms of $\frac{d}{dt}A$ and $\frac{d}{dt}B$.

Problem 5. Consider the equations

$$y_1 = x_1 + x_1 x_2 x_3\,, \qquad y_2 = x_2 + x_1 x_3\,, \qquad y_3 = x_3 + 2x_1 + 3x_3^2$$

(i) Verify that $x = y = (0, 0, 0)$ is a solution of those equations.
(ii) Use the inverse function theorem to show that one can solve the equations uniquely for x provided y is close enough to zero. *Hint*: think of $y = f(x)$ for a differentiable function (which one) on $\mathbb{R}^3$; check $f(0) = 0$; now try to find an argument - using a Theorem from class - why one has f^{-1} (at least near $y = 0$).

Problem 6. Consider the equation in 3 variables

$$x_1 + x_2 - y + \sin(x_1 x_2 y) = 0$$

(i) Verify that $(x_1, x_2, y) = (0, 0, 0)$ solves this equation.
(ii) Apply the implicit function theorem to show that one can solve for y explicitly in terms of (x_1, x_2) provided (x_1, x_2) is near $(0, 0)$, that is, show

that there exists a smooth function $y = g(x_1, x_2)$ for (x_1, x_2) near $(0, 0)$ so that

$$x_1 + x_2 - g(x_1, x_2) + \sin(x_1 x_2 g(x_1, x_2)) = 0$$

(iii) Calculate $Dg(x_1, x_2)$.

Problem 7. Is the equation

$$\cos(xy) + e^{x+y} = 1$$

solvable for y as a function of x near $(x, y) = (0, 0)$?