Homework 5, Differential Geometry
due 3/3/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. We need some basic linear ODE results. Let $A\colon I \to \mathbf{gl}(n,\mathbb{R})$ be a smooth function where $I \subset \mathbb{R}$ is an interval and $\mathbf{gl}(n,\mathbb{R})$ denotes the vector space of all $n \times n$ matrices.

(i) If $F\colon I \to \mathbf{gl}(n,\mathbb{R})$ satisfies the matrix ODE $F' = FA$, then $\det F$ satisfies the scalar ODE $(\det F)' = \operatorname{tr} A \det F$. Here $\operatorname{tr} B$ denotes the trace (the sum of the diagonal elements) of an $n \times n$ matrix B.

(ii) If $F\colon I \to \mathbf{gl}(n,\mathbb{R})$ satisfies the matrix ODE $F' = FA$ and for some $t_0 \in I$ we have $F(t_0) \in \mathbf{GL}(n,\mathbb{R})$, where $\mathbf{GL}(n,\mathbb{R})$ denotes the group of invertible matrices, then $F\colon I \to \mathbf{GL}(n,\mathbb{R})$.

(iii) If $F\colon I \to \mathbf{gl}(n,\mathbb{R})$ satisfies the matrix ODE $F' = FA$ and $\operatorname{tr} A = 0$ then $\det F(t)$ is constant in t. In particular, if $\det F(t_0) = 1$ then $\det F(t) = 1$ for all $t \in I$.

(iv) If $F\colon I \to \mathbf{gl}(n,\mathbb{R})$ satisfies the matrix ODE $F' = FA$ and $A\colon I \to \mathbf{so}(n,\mathbb{R})$ takes values in skew-symmetric matrices, then $F(t)F^T(t)$ is constant in t. In particular, if $F(t_0) \in \mathbf{O}(n,\mathbb{R})$ then $F\colon I \to \mathbf{O}(n,\mathbb{R})$. The analogous statement holds for $F(t_0) \in \mathbf{SO}(n,\mathbb{R})$.

We assume from ODE theory that given a smooth $A\colon I \to \mathbf{gl}(n,\mathbb{R})$ there exists a unique smooth solution $F\colon I \to \mathbf{gl}(n,\mathbb{R})$, defined on the same interval I on which A is defined, of the initial value problem $F' = FA$ and $F(t_0) = F_0 \in \mathbf{gl}(n,\mathbb{R})$ given.

(i) Show that two solutions $F_i\colon I \to \mathbf{GL}(n,\mathbb{R})$ of the ODE $F' = FA$ satisfy $F_2 = CF_1$ for a constant invertible matrix $C \in \mathbf{GL}(n,\mathbb{R})$.

(ii) Show that for A a constant matrix $F(t) = \exp(tA)$ is a solution of $F' = FA$ where $\exp(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!}$ for $B \in \mathbf{gl}(n,\mathbb{R})$ is the matrix exponential map (show that the sum converges absolutely and uniformly on compact sets of matrices – take as a norm on matrices the Euclidean norm, i.e. the square root of the sum of the squares of the entries; thus one can do term by term differentiation; be careful about the non-commutativity of matrices).

(iii) If $A\colon I \to \mathbf{gl}(n,\mathbb{R})$ is not constant, why is $F(t) = \exp(\int_{t_0}^{t} A(s)ds)$ not solving $F' = FA$, or is it? Explain.

Problem 2. Let $\gamma\colon I \to \mathbb{R}^3$ be an arclength parametrized curve whose image lies in the 2-sphere S^2 , i.e. $||\gamma(t)||^2 = 1$ for all $t \in I$. Consider the "moving basis" $\{T, \gamma \times T, \gamma\}$ where $T = \gamma'$.

(i) Writing the moving basis as a 3×3 matrix $F := (T, \gamma \times T, \gamma)$ (where we think of T and etc. as column vectors) show that $F\colon I \to \mathbf{SO}(3,\mathbb{R})$;

(ii) Define the curvature $\kappa := < T', \gamma \times T >$ in analogy to plane curves (noting that $\gamma \times T$ is normal to T and tangent to the 2-sphere–so $\gamma \times T$ takes the role of JT for plane curves). Calculate $A := F^{-1}F'$ and show that $A\colon I \to \mathbf{so}(3,\mathbb{R})$ takes values in skew-symmetric matrices $\mathbf{so}(3,\mathbb{R})$, depends only on κ, and has a very special form (which one).

(iii) Calculate the curvature κ for the circles C in S^2 obtained by slicing S^2 by the planes $z = c$ with $0 \le c < 1$, and calculate $\int_C \kappa$.

(iv) Let $\kappa\colon I \to \mathbb{R}$ be a smooth function. Show that there is a curve $\gamma\colon I \to S^2$ whose curvature is κ. *Hint*: build the matrix function A out of κ (see (ii)) and consider the linear matrix ODE $F' = FA$; use Problem 1 to solve; take the 3rd column of F as a candidate for γ...

(v) The curve constructed in (iii) is unique up to a rotation of S^2. (The group of rotations $\mathbf{SO}(3,\mathbb{R})$ of $\mathbb{R}^3$ preserve S^2–why?)

Problem 3. The Frenet frame of a curve in $\mathbb{R}^3$. For a regular plane curve (and more generally for a regular curve on a 2-dimensional surface - e.g. the 2-sphere above) we could construct a unique adapted frame F. This is not the case for curves in higher dimensional spaces. Besides the curve being regular we need more conditions to ensure the existence of a unique adapted frame, which then will give invariants of the curve, which in turn reconstruct the curve up to Euclidean motions. Let $\gamma\colon I \to \mathbb{R}^3$ be an arclength parametrized curve. Then $T = \gamma'$ has unit length.

(i) Show that $< \gamma'', T >= 0$. Thus, provided that γ'' is nowhere vanishing, we can define $N := \gamma''/||\gamma''||$ and obtain a moving basis $\{T, N, T \times N\}$. A regular space curve for which γ'' is nowhere vanishing is called a *Frenet curve.*

(ii) Let γ be an arclength parametrized Frenet curve. Define the curvature function to be $\kappa := ||\gamma''|| > 0$ and the *torsion* function $\tau :=< T \times N, N' >$. Show that the adapted frame $F = (T, N, T \times N)\colon I \to \mathbf{SO}(3,\mathbb{R})$ and calculate $A = F^{-1}F'$ in terms of κ and τ.

(iii) If γ is am arclength parametrized plane curve, we can regard it a a space curve. Show that this space curve has $\tau \equiv 0$. Also prove the converse: if a space curve has $\tau \equiv 0$ then it lies in a plane in $\mathbb{R}^3$.

(iv) Show that given $\kappa\colon I \to \mathbb{R}$, $\tau(t) > 0$ for all $t \in I$, and $\tau\colon I \to \mathbb{R}$ smooth, there exists a unique (up to Euclidean motion) Frenet curve in $\mathbb{R}^3$ whose curvature and torsion are κ and τ respectively.

(v) Classify the Frenet space curves which have curvature and torsion constant.