Homework 5, Advanced Calculus
due 3/1/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Consider the function $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/||x||$.

(i) Draw the graph of this function for $n = 2$.
(ii) Calculate $Df(x)$.
(iii) Assume a curve $\gamma\colon I \to \mathbb{R}^n \setminus \{0\}$, $I \subset \mathbb{R}$ an interval, satisfies the (2nd order differential) equation $\gamma''(t) = Df(\gamma(t))$ for all $t \in I$. Verify that the function $L\colon I \to \mathbb{R}$ given by

$$L(t) = \tfrac{1}{2}||\gamma'(t)||^2 - f(\gamma(t))$$

is constant, that is, does not depend on t. *Hint:* recall a criterion from Calc I which tells you when a function in one variable is constant.

Problem 2. Let $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = 1/||x||$ (as in the previous problem, but now $n = 2$).

(i) Verify that $\gamma(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$ parametrizes an ellipse. Assuming $a \geq b > 0$ what is the equation of this ellipse, what are the major and minor axes? Draw a picture.
(ii) Calculate the rate of change of f along the ellipse γ. What happens when the ellipse is a circle?
(iii) If the ellipse is not a circle, for which values of $t \in [0, 2\pi]$ is that rate of change maximal/minimal.

Problem 3. Consider the logarithmic spiral $\gamma(t) = e^{bt}(\cos t, \sin t)$ for $t \in \mathbb{R}$.

(i) Draw a picture of this curve.
(ii) Verify that the angle between $\gamma(t)$ and $\gamma'(t)$ is constant, i.e. independent of t. Express that angle in terms of b.
(iii) Verify that for each $t \in \mathbb{R}$ the angle between $\gamma'(t)$ and the tangent vector v to the circle (centered at the origin) at the point $\gamma(t)$ of radius $||\gamma(t)||$ is the same. Draw a picture and indicate which angle this is and relate it to the angle from (ii).
(iv) We now think of geography, maps and sailing: let D be the unit radius closed disk and interpret the radial lines as longitudes and the concentric circles as latitudes. Starting at the equator you sail north in such a way that your angle to all the latitudes you cross is always 30 degrees. What course are you sailing, i.e. determine a formula for the curve γ you are following. Draw a picture.
(v) Assume the temperature $T(x) = 1 - e^{-||x||^2}$ on the unit disc D (draw the graph of T over D). Calculate the rate of change of T per unit distance travelled when you follow your course $\gamma(t)$.