HOMEWORK 5, M 331.2
DUE 10/19/16

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Let $y(t)$ denote the concentration of a substance (drug, hormones etc.) in an organism. We assume that the organism eliminates the substance at a rate proportional to the amount present, and that the substance is periodically administered over time. The ODE for this model is

$$\dot{y}(t) = -k\,y(t) + A + B\cos(\omega t)$$

where k is the elimination rate, A should be interpreted as the average concentration of the substance, B the amplitude and ω is the frequency of administration of the substance.

(i) Find the general solution of the ODE for the case $k = 1/3$, $A = B = 1$ and $\omega = \pi/3$.
(ii) Find the solution with initial condition $y(0) = 0$.
(iii) Which terms in your solution survive (steady state) and which die out for large time.
(iv) One would think that the concentration $y(t)$ is largest around the time when the (externally) administered substance reaches its maximum, that is at the crest of $1 + \cos(\frac{\pi}{3}t)$. Is this so, and if not, when does it reach its maximum?

Problem 2. Calculate the time it takes for a mass of 80 kg dropped from a height of 100 meters to reach the ground, first without friction and then with friction modeled proportional to velocity (assume the drag coefficient is 1).

Problem 3. Your rate of increase (hopefully) over time t of understanding ODEs is proportional (with some learning factor k) to the product of what you already know $Q(t)$ and what you don't know $A - Q(t)$ (assuming there is finite amount A to be known about ODEs).

(i) Write down the differential equation for this model.
(ii) Interpret the equilibrium solutions in practical terms.
(iii) Find the general solution of your ODE.
(iv) At which point does your understanding increase the fastest?
(v) If you start out with hardly any knowledge $Q(0) = 1$, and $A = 10^3$, t is measured in days, $k = 10^{-4}$, how long would it take you to understand at least 75% of ODE theory?

Problem 4. Consider the linear (inhomogeneous) ODE

$$y' - y = 1 + t^2 + \cos t$$

(i) Find all solutions to the homogeneous ODE.
(ii) Find a particular solution of the inhomogeneous ODE.
(iii) Write down all solutions of the ODE.
(iv) Find the solution which satisfies $y(0) = 0$.

Problem 5. Consider the ODE

$$y' = y^2 - 4$$

(i) Draw a slope line picture, indicate the equilibrium solutions and draw a graph of a few more solutions.
(ii) Characterize the equilibria as stable/unstable/semistable.
(iii) Calculate the solution of the ODE satisfying the initial condition $y(0) = -3$ and draw it into your slope line picture.
(iv) Is there a vertical asymptote for your solution, in other words, does the solution tend to infinity in finite time?

Problem 6. Find the general solution to the ODE

$$y' + \tfrac{3}{t}y = t^3 .$$

Problem 7. Solve the initial value problem

$$\frac{dy}{dt} = 2y^2 + ty^2$$
$$y(0) = -1/2$$

and determine whether, and if where, the solution has a vertical asymptote.

Problem 8. Find the solution of

$$y' + 4y = e^{-4t} + t^2$$

with initial condition $y(0) = 0$.

Problem 9. Find the general solution of

$$y' - y = t\cos(t)$$

Problem 10. Find the solution of

$$y' = 2y + e^{2t}\ln(t)$$

with initial condition $y(1) = 0$.